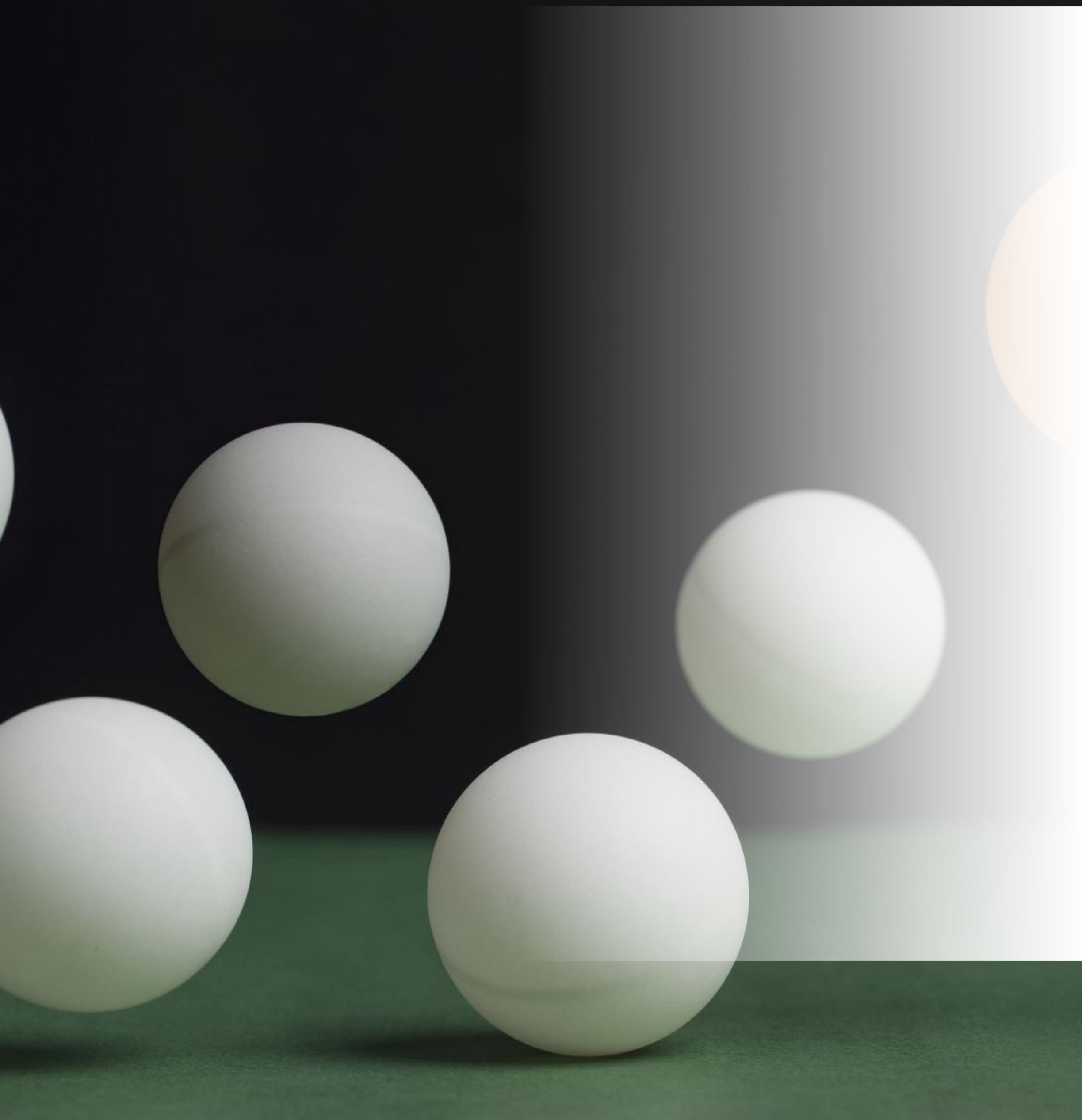
Seminar Talk GSI, September 11, 2024



Three-body amplitudes for the analysis of lattice data and experiment

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With slide material from
Y. Feng and M. Mai

Overview

Review 2B-lattice: [Briceno]
Reviews 3B-lattice: [Hansen] [Mai]
Review hadron resonances: [Mai]

Key publications Finite-Volume Unitary (FVU) approach:
- Three-body unitarity [Mai/JPAC]
- Three-body unitarity finite volume [Mai]
- a_1 in finite volume & results from lQCD [Mai]

Talk outline:
- 3-body unitarity
- a_1 in infinite volume
- $3\pi^+$, a_1 in finite volume
- Recent extensions: channel space & applications

(Design: Y. Feng, based on TV series logo)

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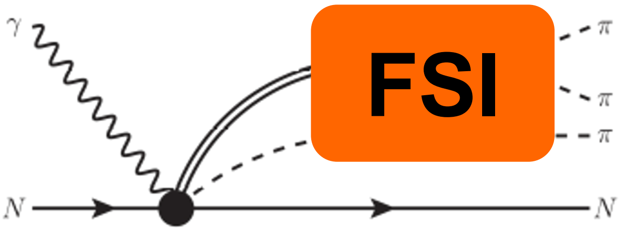
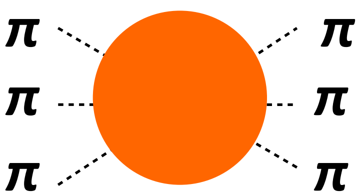
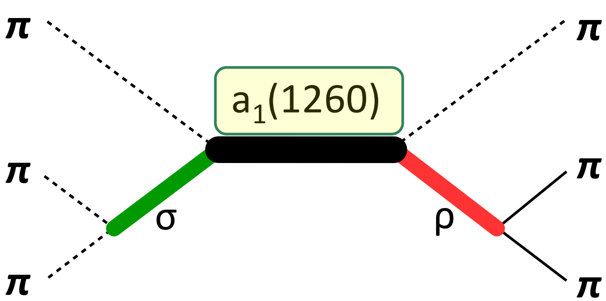
Three-body aspects: $\pi\pi N$ vs. $\pi\pi\pi$

Light mesons

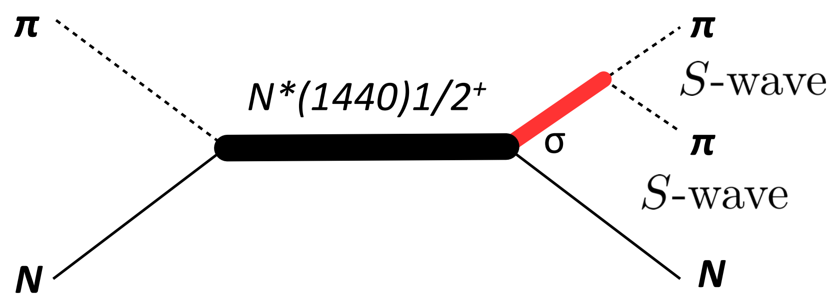
- COMPASS @ CERN: $\pi_1(1600)$ discovery
- GlueX @ Jlab in search of hybrids and exotics,
- Finite volume spectrum from lattice QCD:
  Lang (2014), Woss [HadronSpectrum] (2018)
  Hörz (2019), Culver (2020, 21,…), Fischer (2020),
  Hansen/HadSpec (2020),…







Light baryons



- Roper resonance is debated for ~50 years in experiment.
- 1st calculation w. meson-baryon operators on the lattice: **Lang et al. (2017)**

How Many Resonances Decay to 3 Particles?

[PDG]

Just one sub-family of resonances (N*):

N Baryons (S = 0, I = 1/2)			
p	PDF	N(1900) 3/2+	PDF
n	PDF	N(2000) 5/2+	PDF
N(1440) 1/2+	PDF	N(1990) 7/2+	PDF
N(1520) 3/2−	PDF	N(2040) 3/2+	PDF
N(1535) 1/2−	PDF	N(2060) 5/2−	PDF
N(1650) 1/2−	PDF	N(2100) 1/2+	PDF
N(1675) 5/2−	PDF	N(2120) 3/2−	PDF
N(1680) 5/2+	PDF	N(2190) 7/2−	PDF
N(1700) 3/2−	PDF	N(2220) 9/2+	PDF
N(1710) 1/2+	PDF	N(2250) 9/2−	PDF
N(1720) 3/2+	PDF	N(2300) 1/2+	PDF
N(1860) 5/2+	PDF	N(2570) 5/2−	PDF
N(1875) 3/2−	PDF	N(2600) 11/2−	PDF
N(1880) 1/2+	PDF	N(2700) 13/2+	PDF
N(1895) 1/2−	PDF	N(3000 Region)	PDF

p: stable

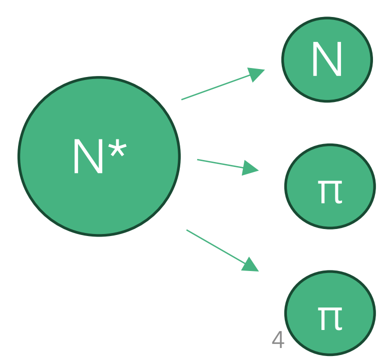
n: “stable”

N(1900) 3/2+, N(2000) 5/2+: ≥3-body decays

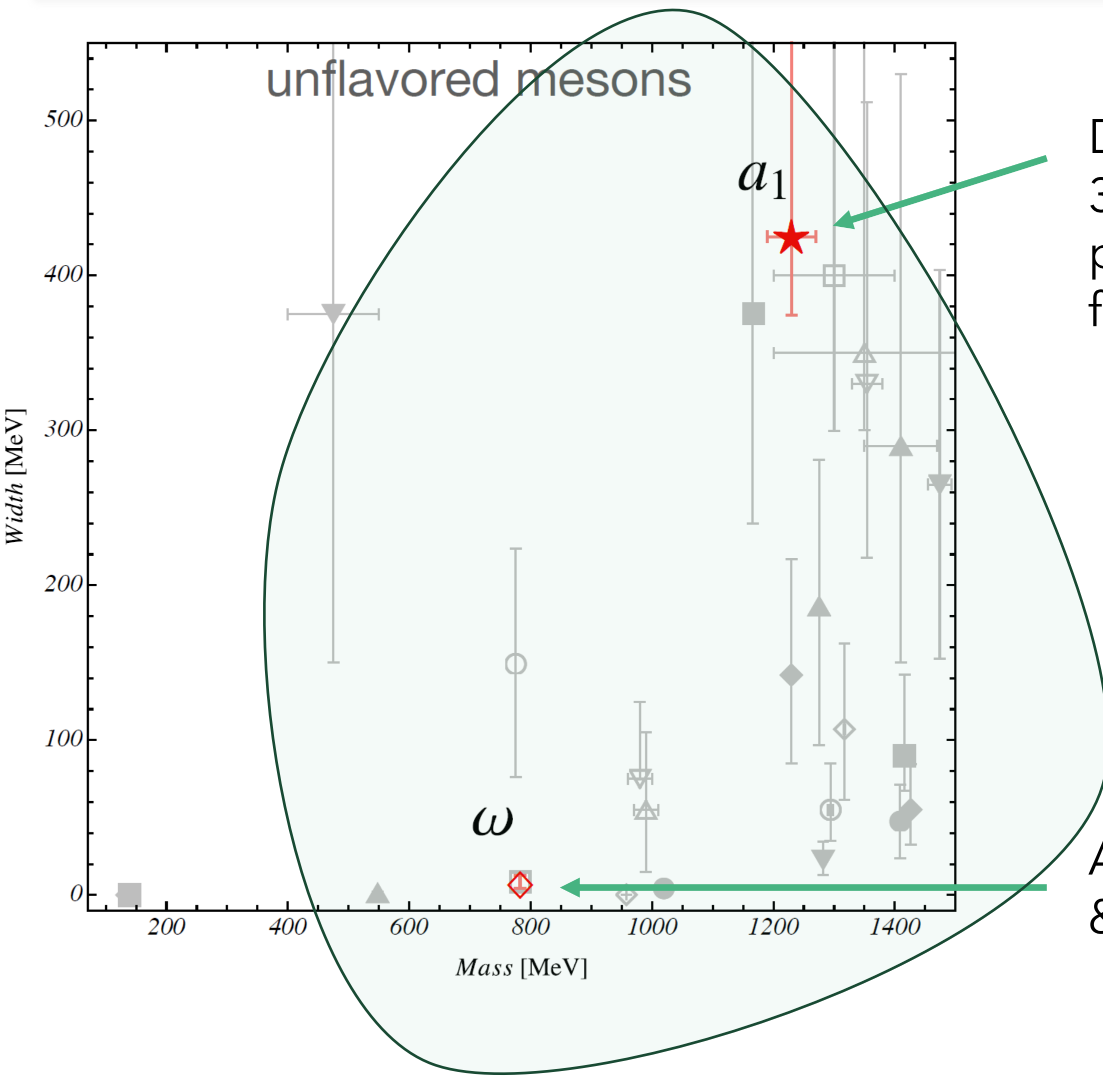
All other N*: ≥ 3-body decays

Number of states With only two-body decays in this sub-family:

0



>2-body meson decays



Decays almost 100% to 3 pions (decay to 2 pseudoscalar mesons forbidden)

Almost stable; lattice results & chiral extrapolation [Yan, Mai et al., 2407.16659]

Three-body unitarity with isobars *

[Mai 2017]

$$\langle q_1,q_2,q_3|(\hat{T}-\hat{T}^\dagger)|p_1,p_2,p_3\rangle = i\int_P \langle q_1,q_2,q_3|\hat{T}^\dagger|k_1,k_2,k_3\rangle\langle k_1,k_2,k_3|\hat{T}|p_1,p_2,p_3\rangle$$

$$\times \prod_{\ell=1}^{3}\left[\frac{\mathrm{d}^4k_\ell}{(2\pi)^4}(2\pi)\boxed{\delta^+(k_\ell^2-m^2)}\right](2\pi)^4\delta^4\left(P-\sum_{\ell=1}^{3}k_\ell\right)$$

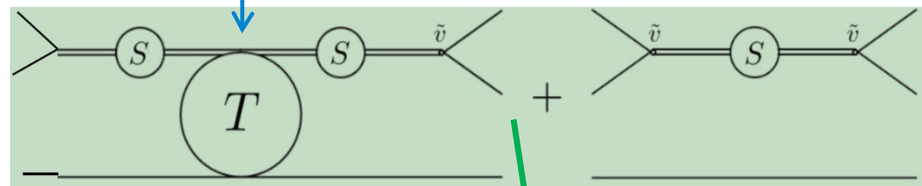
delta function sets all intermediate particles on-shell

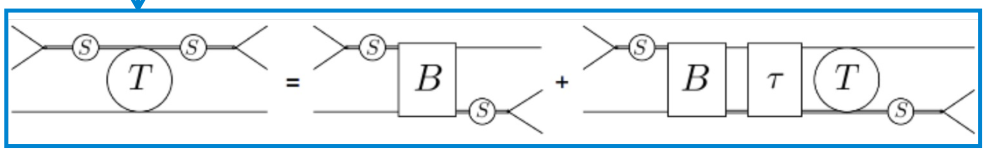
Idea: To construct a 3B amplitude, start directly from unitarity (based on ideas of 60's); match a general amplitude to it

* "Isobar" stands for two-body sub-amplitude which can be resonant or not; can be matched to CHPT expansion to one loop if desired. Isobars are re-parametrizations of full 2-body amplitudes [Bedaque] [Hammer]

Three-body unitarity

$$\langle q_1, q_2, q_3|(\hat{T} - \hat{T}^\dagger)|p_1, p_2, p_3\rangle \quad = \quad i\int_P \langle q_1, q_2, q_3|\hat{T}^\dagger|k_1, k_2, k_3\rangle\langle k_1, k_2, k_3|\hat{T}|p_1, p_2, p_3\rangle$$



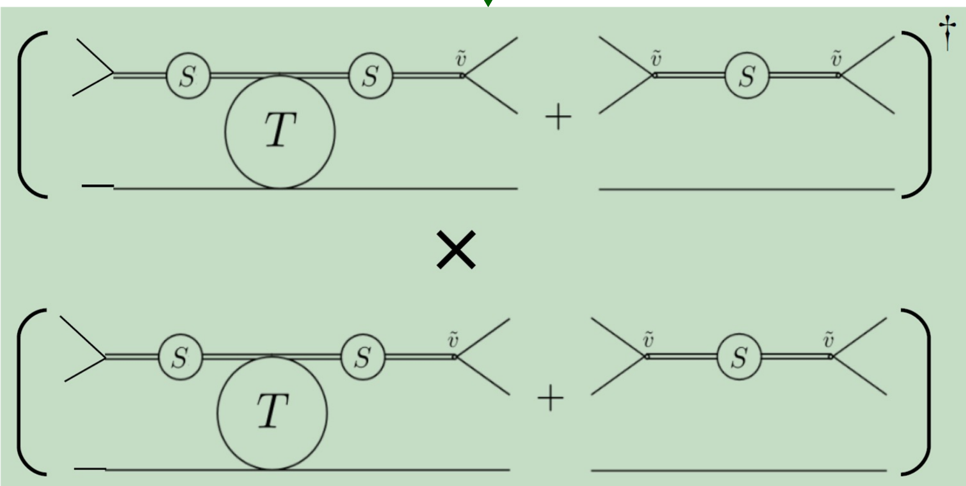


General Bethe-Salpeter Ansatz for the isobar-spectator interaction
→ **B & τ** are unknown functions to be obtained by matching To right-hand side.

The three-pion state is populated by first combining two states to an "isobar", and then adding the third "spectator"

Three-body unitarity

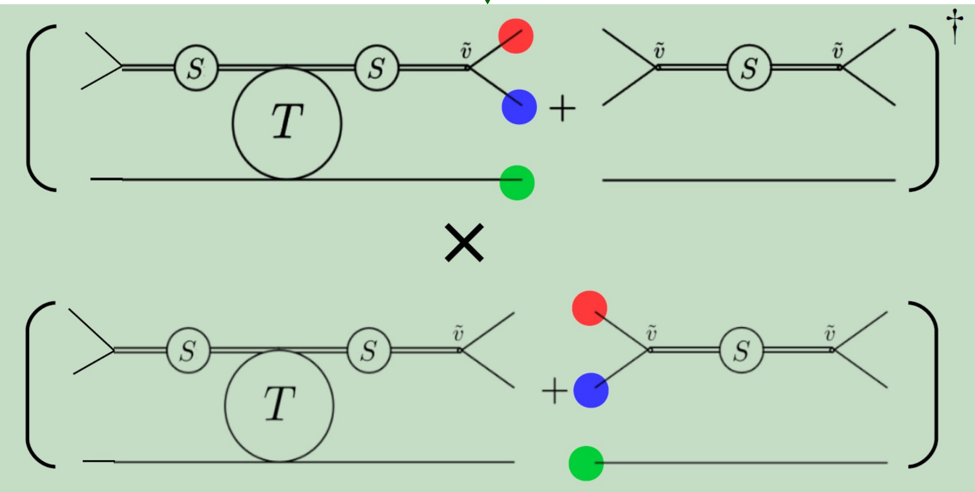
$$\langle q_1, q_2, q_3|(\hat{T}-\hat{T}^\dagger)|p_1, p_2, p_3\rangle = i\int_P \langle q_1, q_2, q_3|\hat{T}^\dagger|k_1, k_2, k_3\rangle\langle k_1, k_2, k_3|\hat{T}|p_1, p_2, p_3\rangle$$

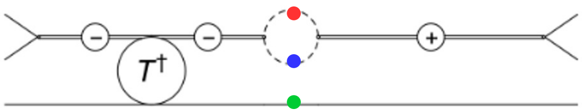


General connected-disconnected structure

Three-body unitarity

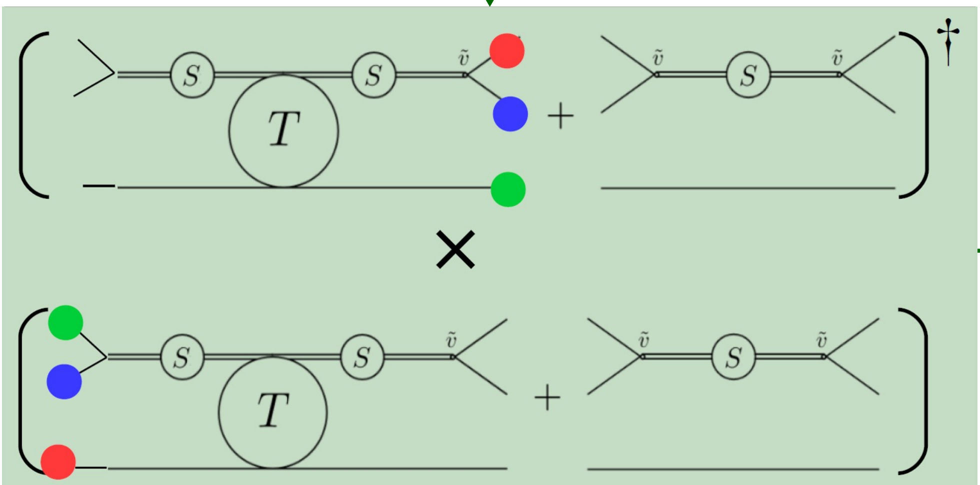
$$\langle q_1, q_2, q_3|(\hat{T}-\hat{T}^\dagger)|p_1, p_2, p_3\rangle \quad = \quad i\int_P \langle q_1, q_2, q_3|\hat{T}^\dagger|k_1, k_2, k_3\rangle\langle k_1, k_2, k_3|\hat{T}|p_1, p_2, p_3\rangle$$

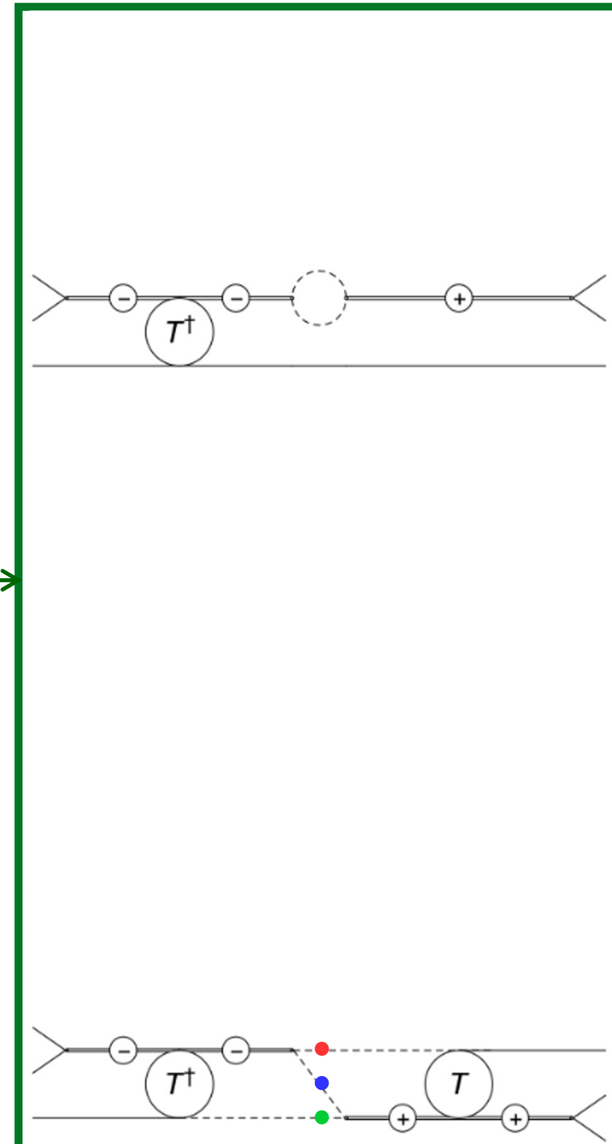




Three-body unitarity

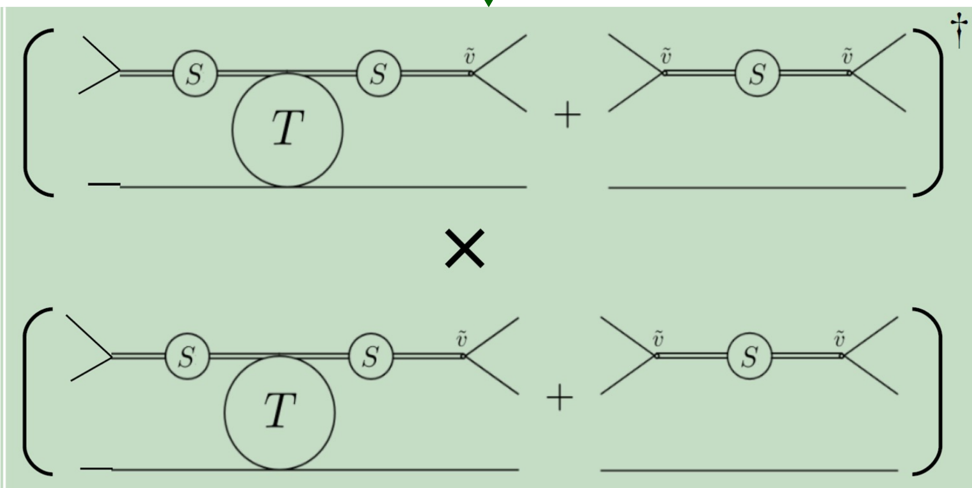
$$\langle q_1, q_2, q_3 | (\hat{T} - \hat{T}^\dagger) | p_1, p_2, p_3 \rangle = i \int_P \langle q_1, q_2, q_3 | \hat{T}^\dagger | k_1, k_2, k_3 \rangle \langle k_1, k_2, k_3 | \hat{T} | p_1, p_2, p_3 \rangle$$



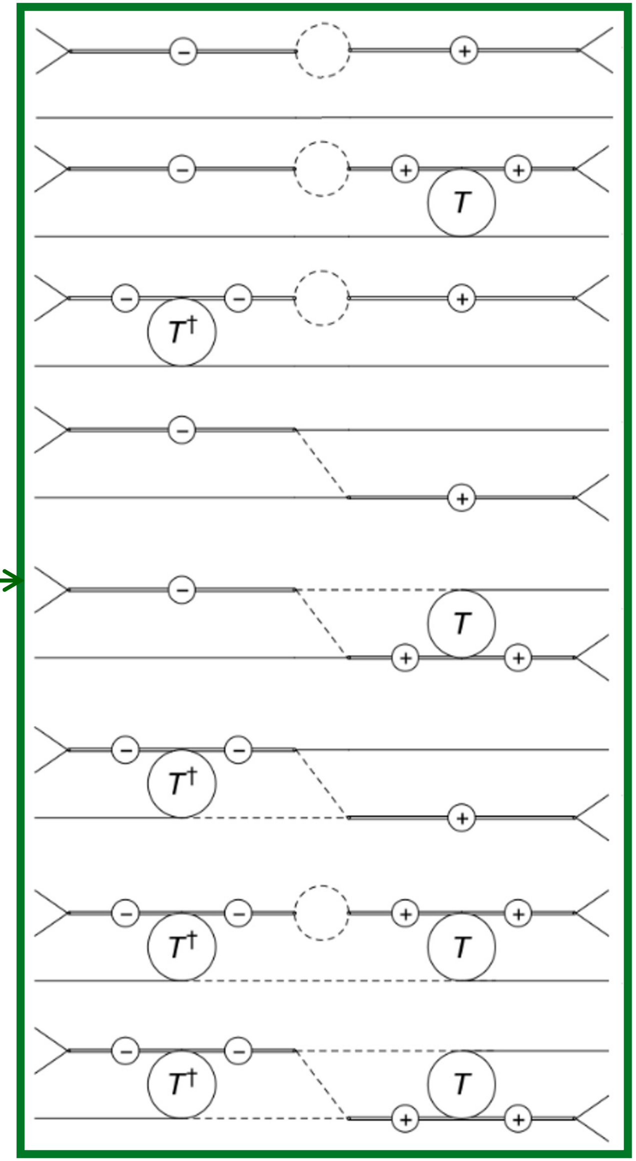


Three-body unitarity

$$\langle q_1,q_2,q_3|(\hat{T}-\hat{T}^\dagger)|p_1,p_2,p_3\rangle = i\int_P \langle q_1,q_2,q_3|\hat{T}^\dagger|k_1,k_2,k_3\rangle\langle k_1,k_2,k_3|\hat{T}|p_1,p_2,p_3\rangle$$



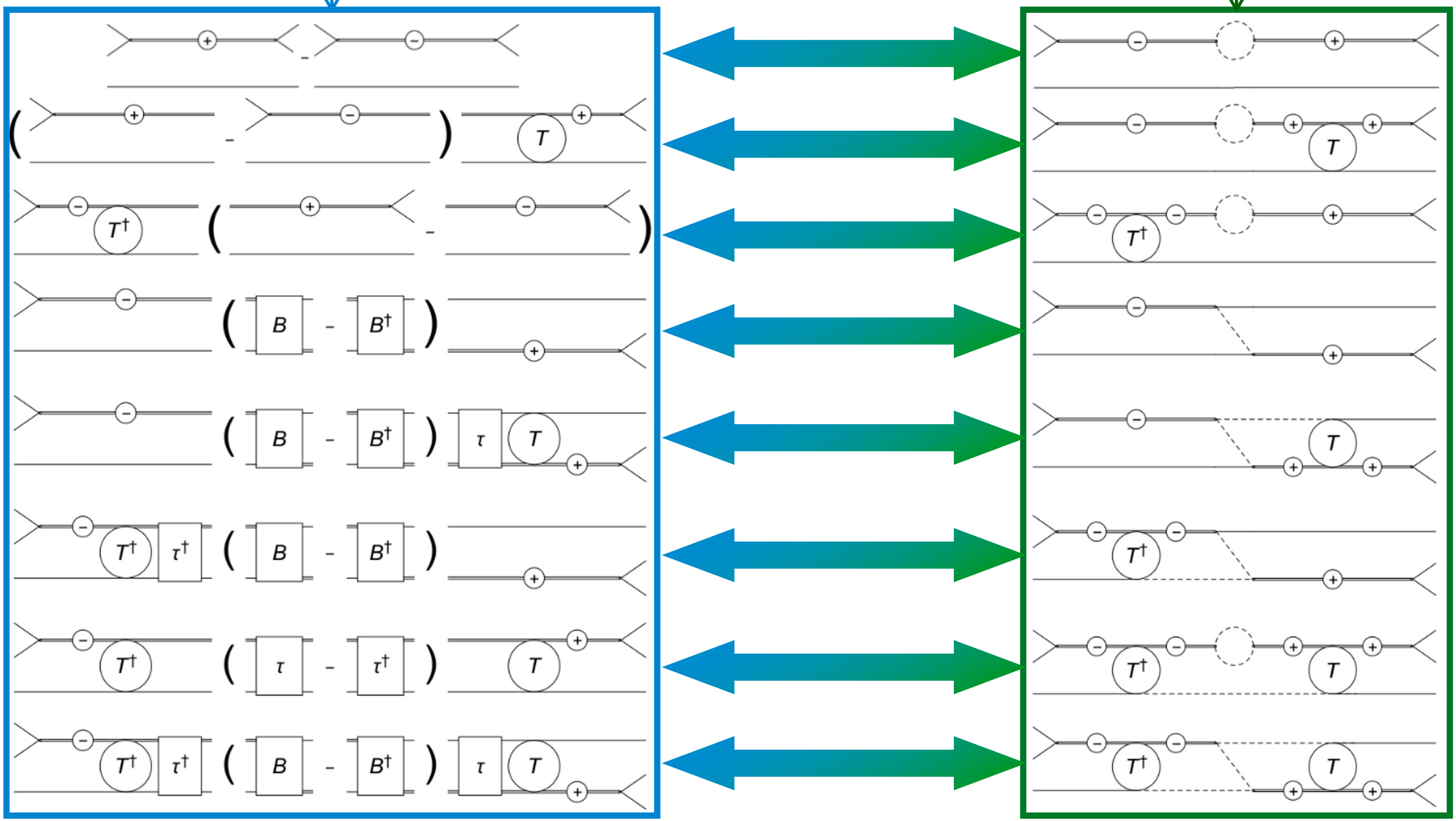
8 topologies



Three-body unitarity

$$\langle q_1,q_2,q_3|(\hat{T}-\hat{T}^\dagger)|p_1,p_2,p_3\rangle \;=\; i\int_P \langle q_1,q_2,q_3|\hat{T}^\dagger|k_1,k_2,k_3\rangle\langle k_1,k_2,k_3|\hat{T}|p_1,p_2,p_3\rangle$$

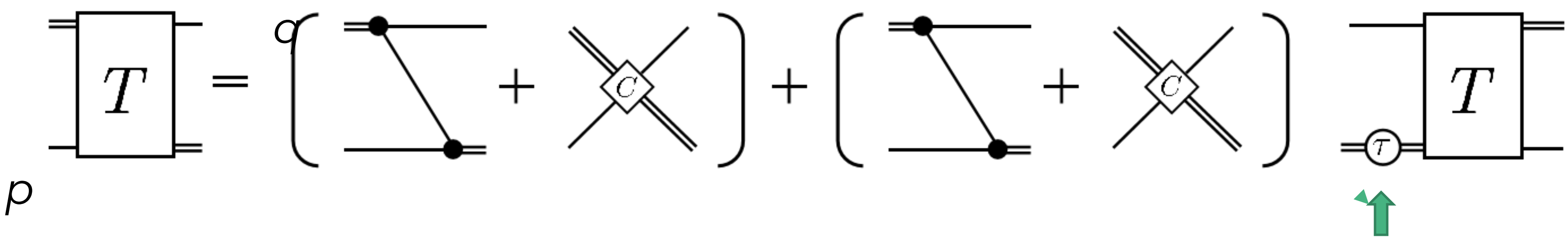
General BSE (T=B+BτT)



Resulting scattering equation

- 4-dim BSE becomes 3-dim by putting spectator on-shell (choice)

$$\langle q|\, T(s)\, |p\rangle = \langle q|\, B(s)\, |p\rangle + \langle q|\, C(s)\, |p\rangle + \int \frac{d^4k}{(2\pi)^4} \langle q|\, (B(s)+C(s))\, |k\rangle\, \tau(\sigma(k))\, \langle k|\, T(s)\, |p\rangle \ ,$$



Exchange:

- Complex
- Required by unitarity

Contact term:

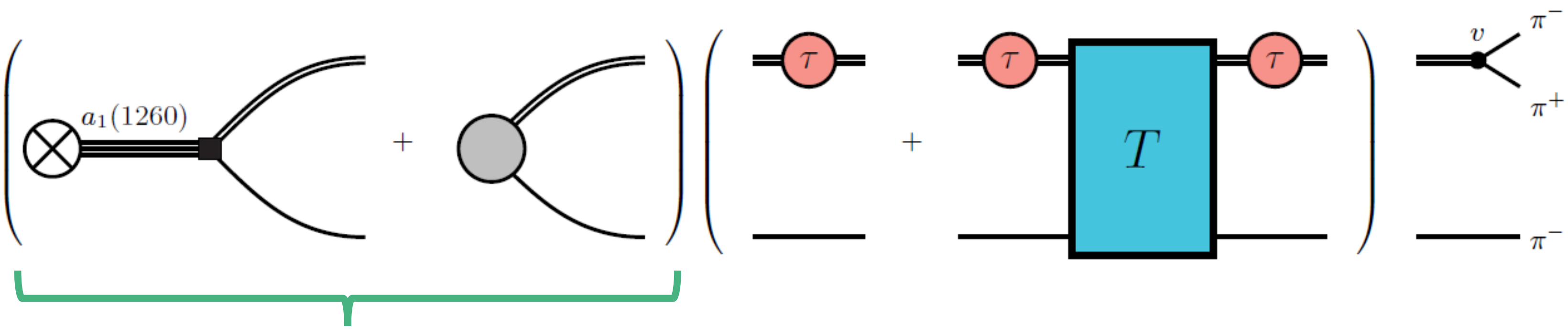
- Does not destroy unitarity
- Free parametrization: fit to data

Isobar-spectator Green's functions

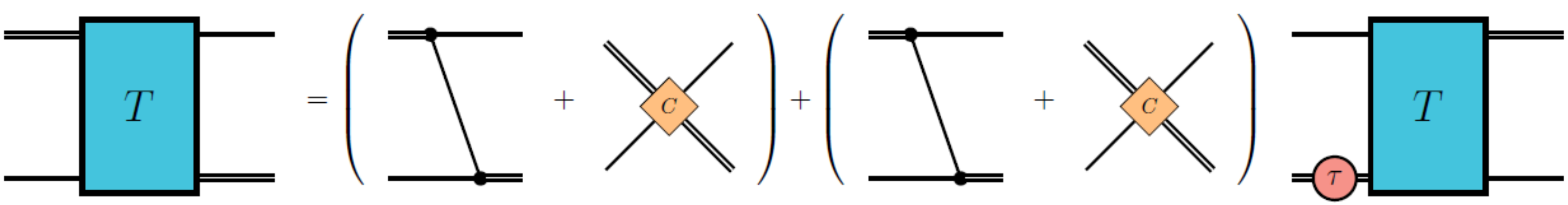
The a_1(1260) and its Dalitz plots

[Sadasivan 2020]

- Disconnected and connected decays for three-body untarity



Additional fit parameters

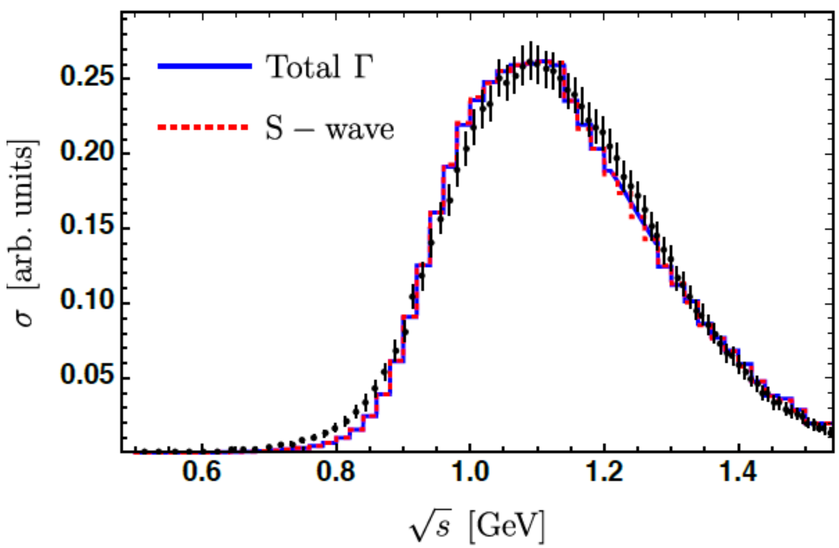


Fitting the lineshape & predicting Dalitz plots

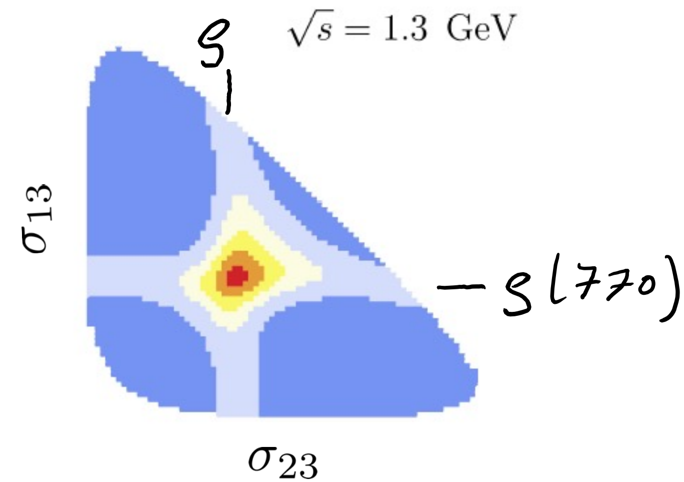
[Sadasivan 2020]

- One can have $\pi\rho$ in S- and D-wave coupled channels
- Fit contact terms to the lineshape from Experiment (ALEPH)

$a_1 \to \pi^-\pi^-\pi^+$ (symmetrize π^-'s!)



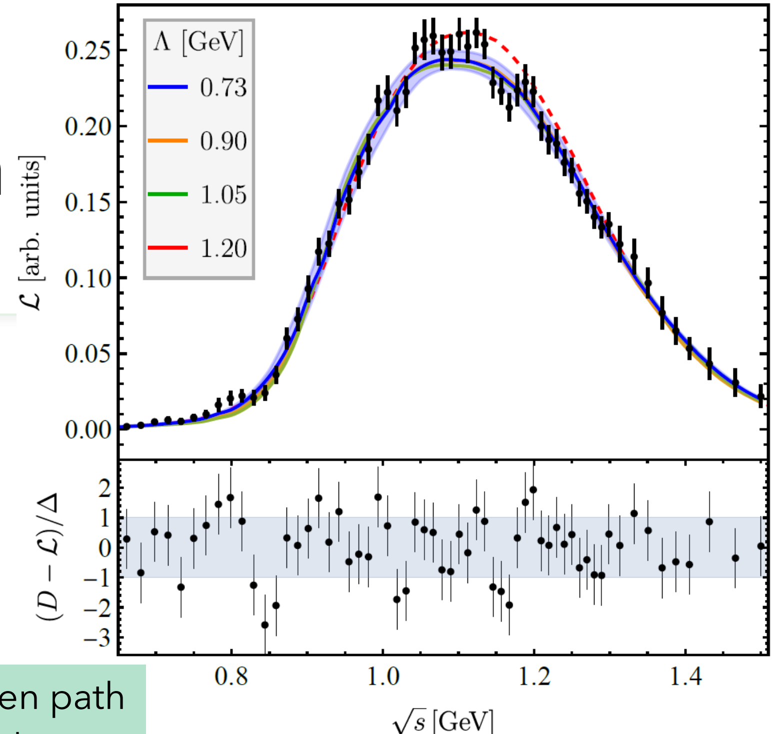
predict →

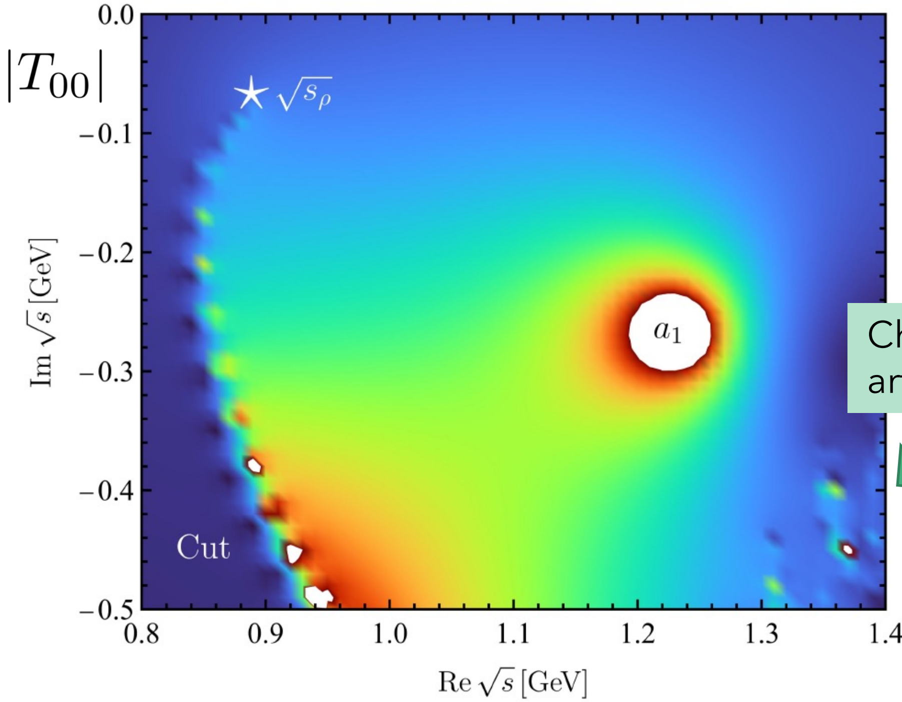


Where is the resonance pole in s? Pole positions & residues are reaction-independent characteristics of resonance mass, width, and branching ratios

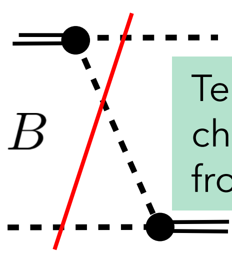
Result: Pole position

Technicalities analytic cont.: contour deformation } [Sadasivan (2021)] [Doering (2009)]



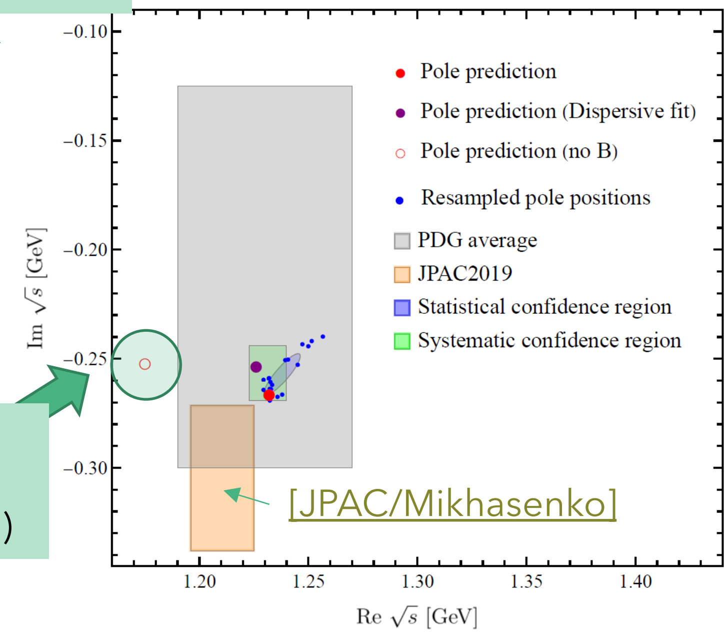


Chosen path artifacts



Technical challenges from 3B cuts

If the B-term is neglected + refit (unitarity violated)



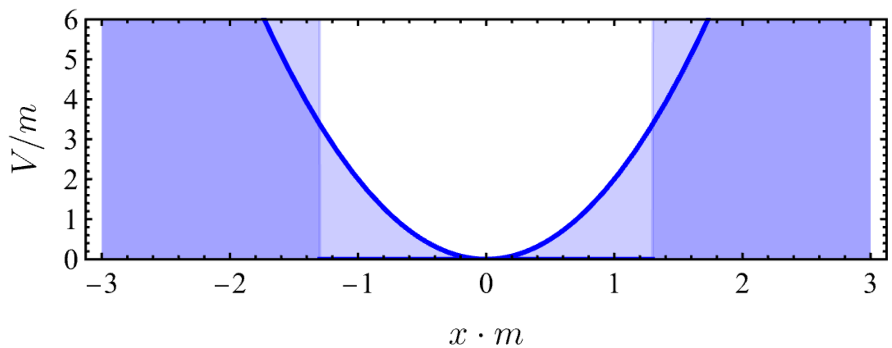
[JPAC/Mikhasenko]

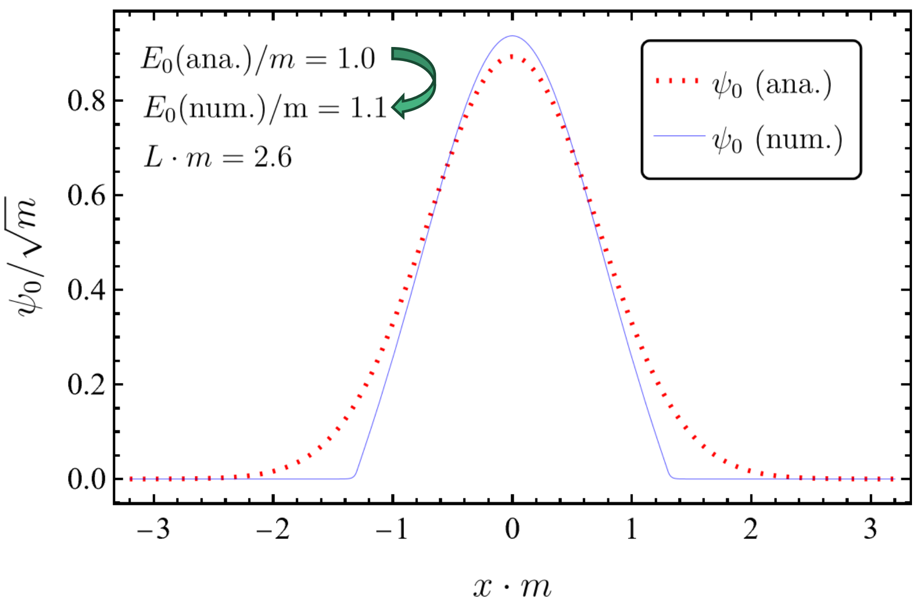
See also: S. M. Dawid [2303.04394 [nucl-th]]

Finite-Volume Effects

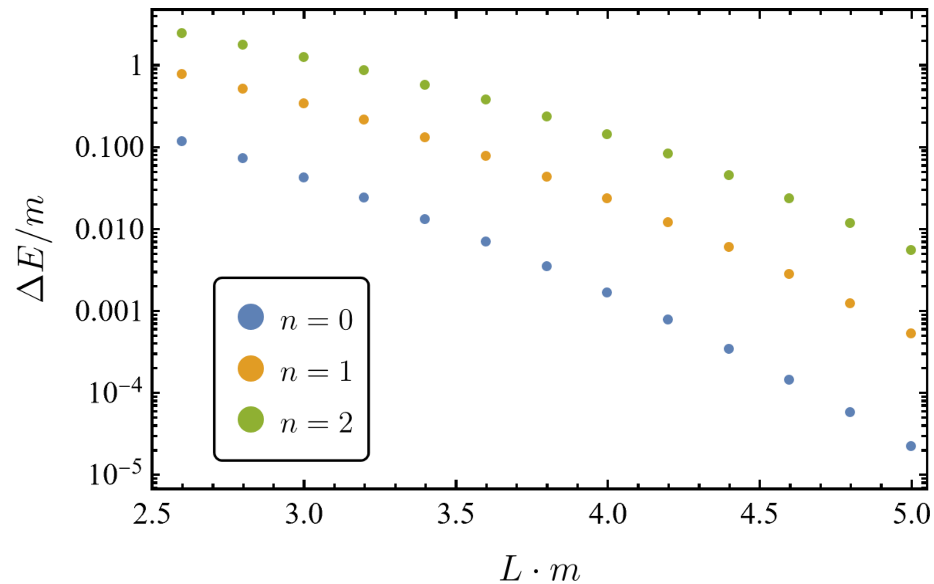
A wave function is squeezed into a finite volume

[https://blogs.gwu.edu/doring/]





Distortion of the energy spectrum as function of box size



Periodic Boundary Conditions

$$\Psi(\vec{x}) \stackrel{!}{=} \Psi(\vec{x} + \hat{\mathbf{e}}_i\, L) = \exp(i\, L\, q_i)\, \Psi(\vec{x})$$

$$q_i = \frac{2\pi}{L} n_i, \quad n_i \in \mathbb{Z}$$

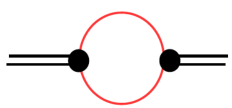
only discrete momenta allowed

Lattice QCD: Finite-volume unitarity (FVU)

Two-body unitarity

↓

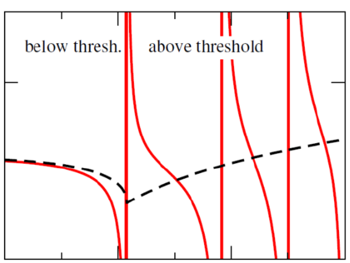
On-shell condition



$\int \to \sum$

↓

Imaginary parts



Infinite → Fin. Vol

↓

Power-law fin-vol. effects

↓

Lüscher

$$p\cot\delta(p) = -8\pi\sqrt{s}\left(\tilde{G}(E) - \mathrm{Re}\,G(E)\right)$$

.

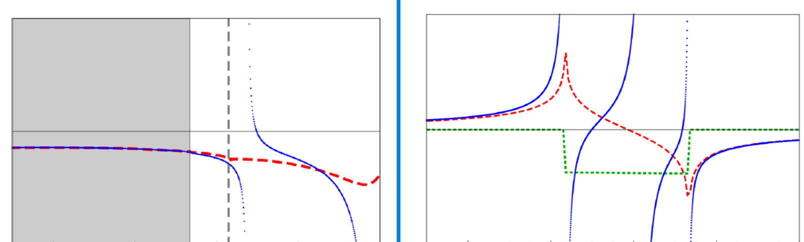
Three-body unitarity

↓

On-shell condition

↓

Imaginary parts



↓

Power-law fin-vol. effects

↓

Quantization Condition

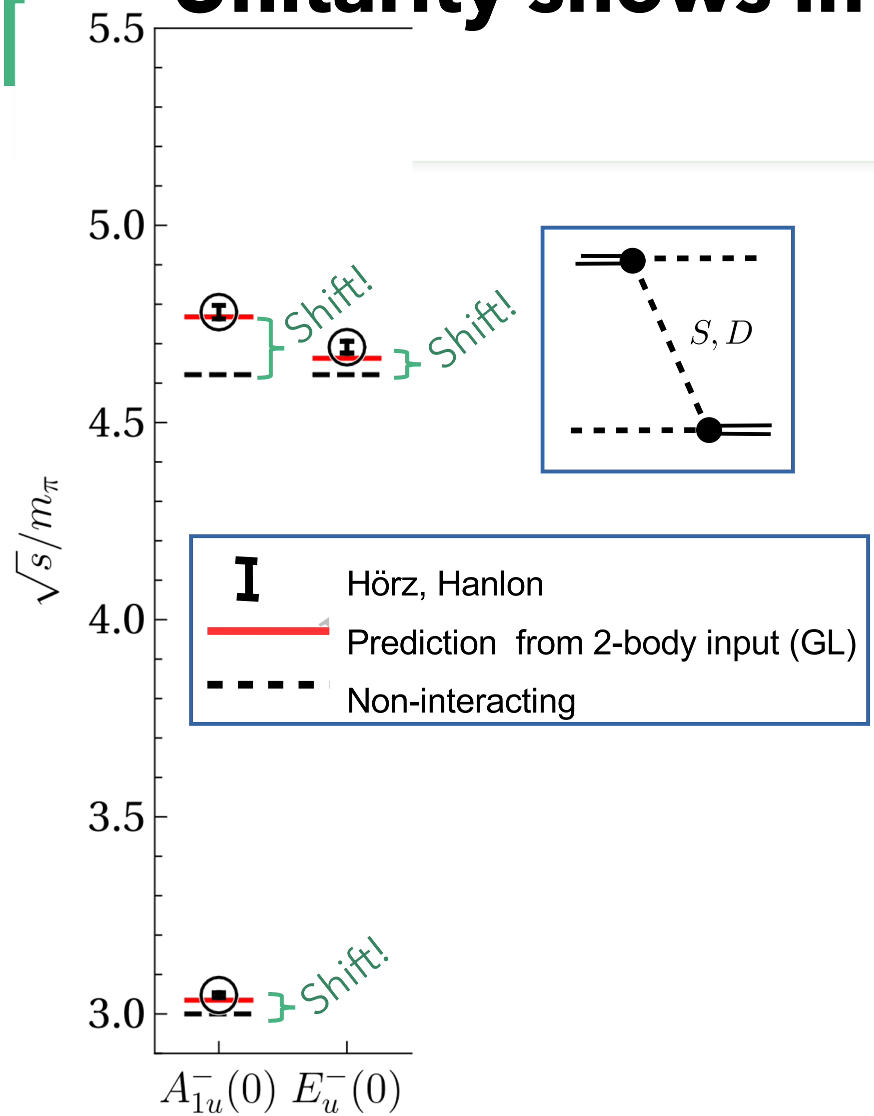
$$\mathrm{Det}\left(\mathbf{B}^{\Gamma ss'}_{\mathbf{uu'}}(\mathbf{W^2}) + \frac{\mathbf{2E_s\,L^3}}{\vartheta(\mathbf{s})}\tau_{\mathbf{s}}(\mathbf{W^2})^{-1}\delta_{\mathbf{ss'}}\delta_{\mathbf{uu'}}\right) = \mathbf{0}$$

.

[Review Mai et al.]

Unitarity shows in FV spectrum: $3\pi^+$

[GWQCD/Culver 2019]



S **D** (lowest participating wave)

- D-wave prediction qualitatively good
  - → Relative/absolute strength between S- and D-wave matched
  - → Consequence that 3-body interaction dominated by exchange
  - → Consequence of 3-body Unitarity

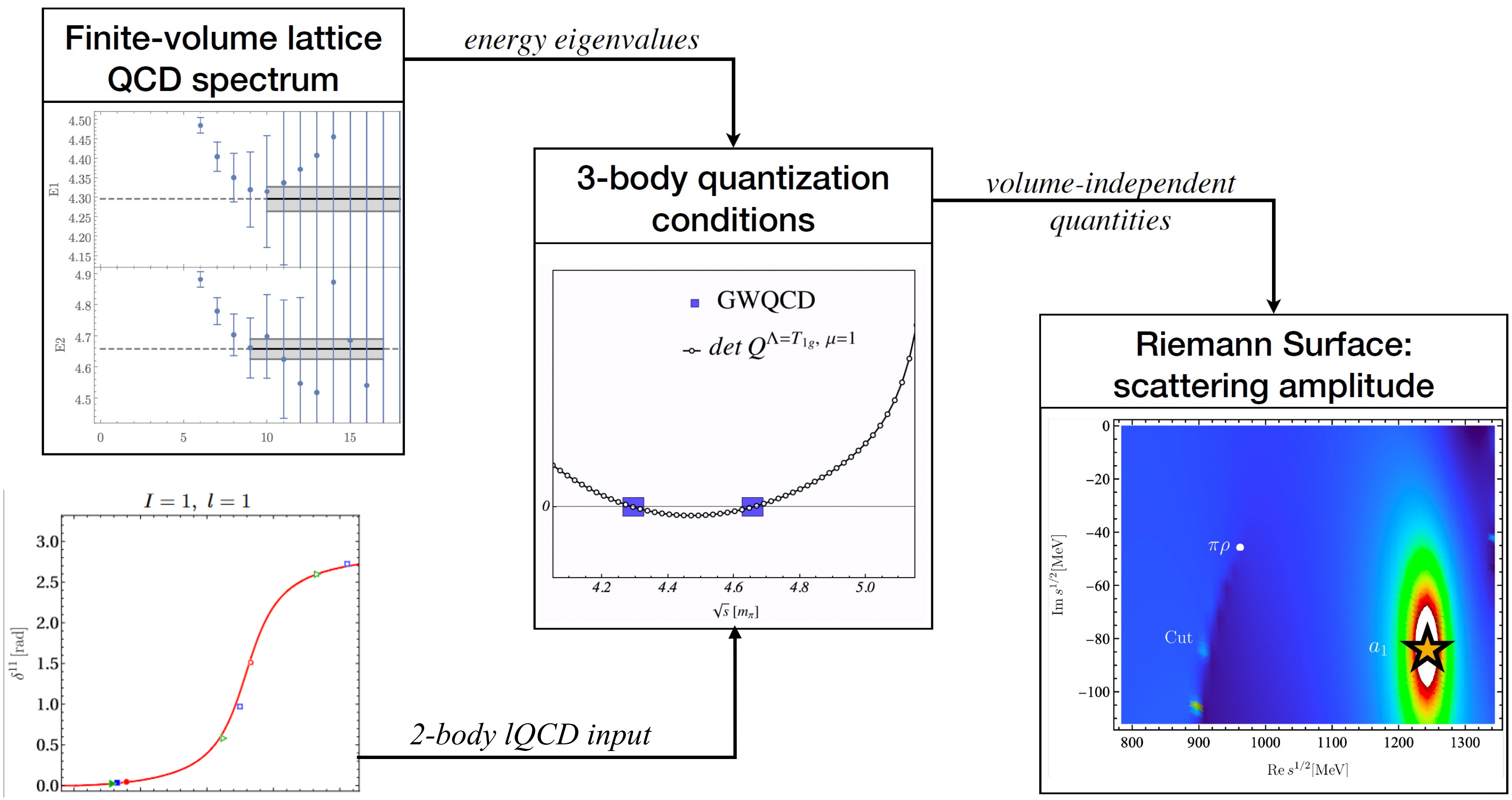
- Three-body unitarity directly visible in the eigenvalue spectrum of lattice QCD

- Many additional levels, including boosts (not shown)

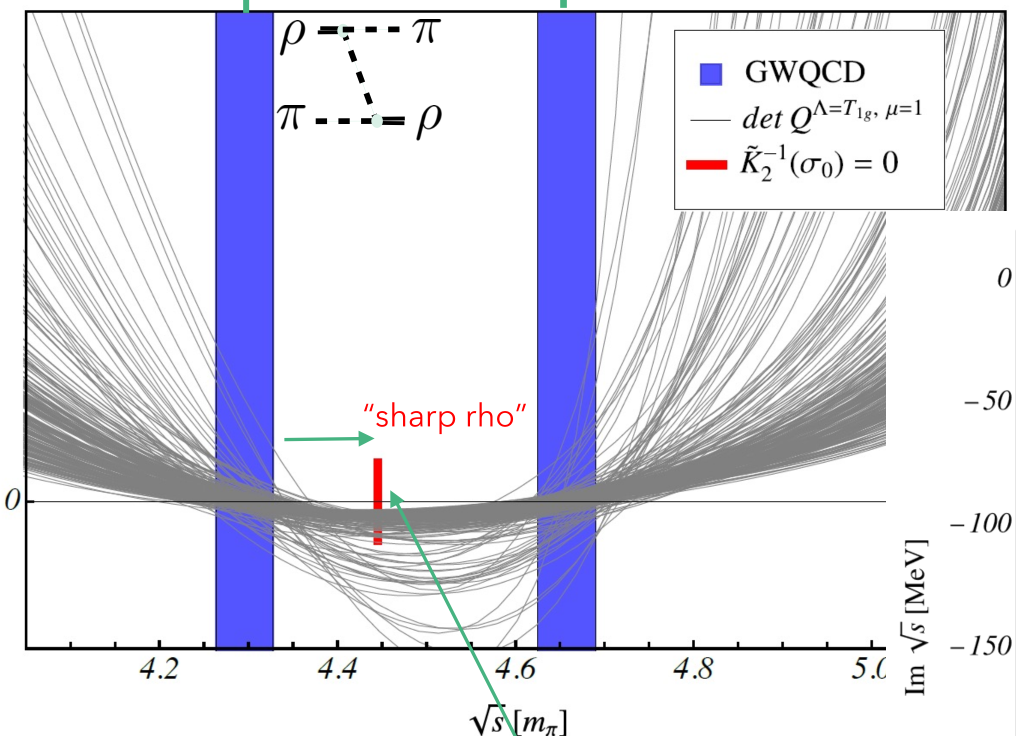
Extraction of $a_1(1260)$ from lQCD

[Mai/MD/GWQCD, PRL 2021]

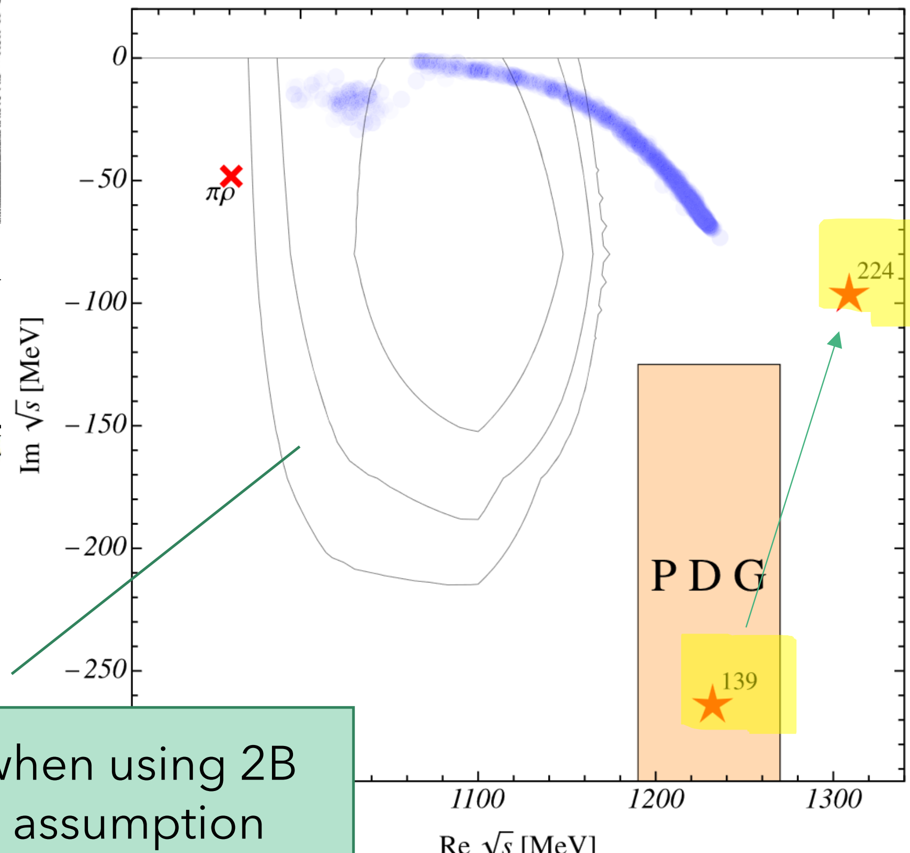
- First-ever three-body resonance from 1st principles (with explicit three-body dynamics).



Results – overview



$$0 = \det\left[B(s) + C(s) - E_L\left(\tilde{K}_2^{-1}(s) - \Sigma_2^L(s)\right)\right]_{(\lambda'\lambda)\atop(\boldsymbol{p}'\boldsymbol{p})}$$



PDF for pole when using 2B Breit-Wigner assumption

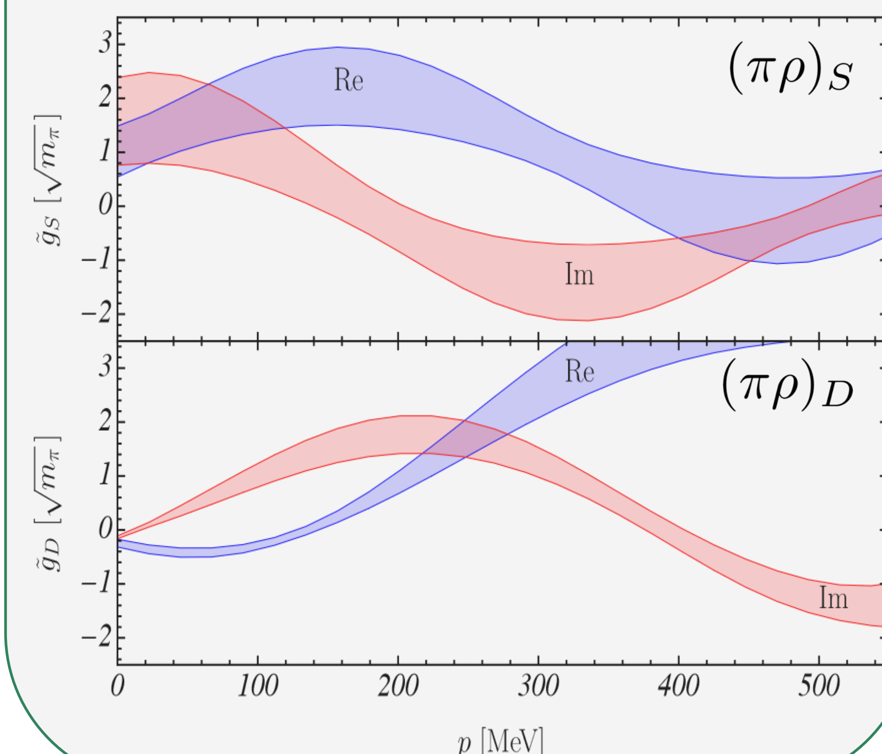
Branching ratios

- Calculate the residue at the pole:

$$\mathrm{Res}(T^c_{\ell'\ell}(\sqrt{s})) = \tilde{g}_{\ell'}\tilde{g}_\ell$$

- This result is not as reliable as pole position/existence of a_1
- More energy eigenvalues needed to better pin down the decay channels

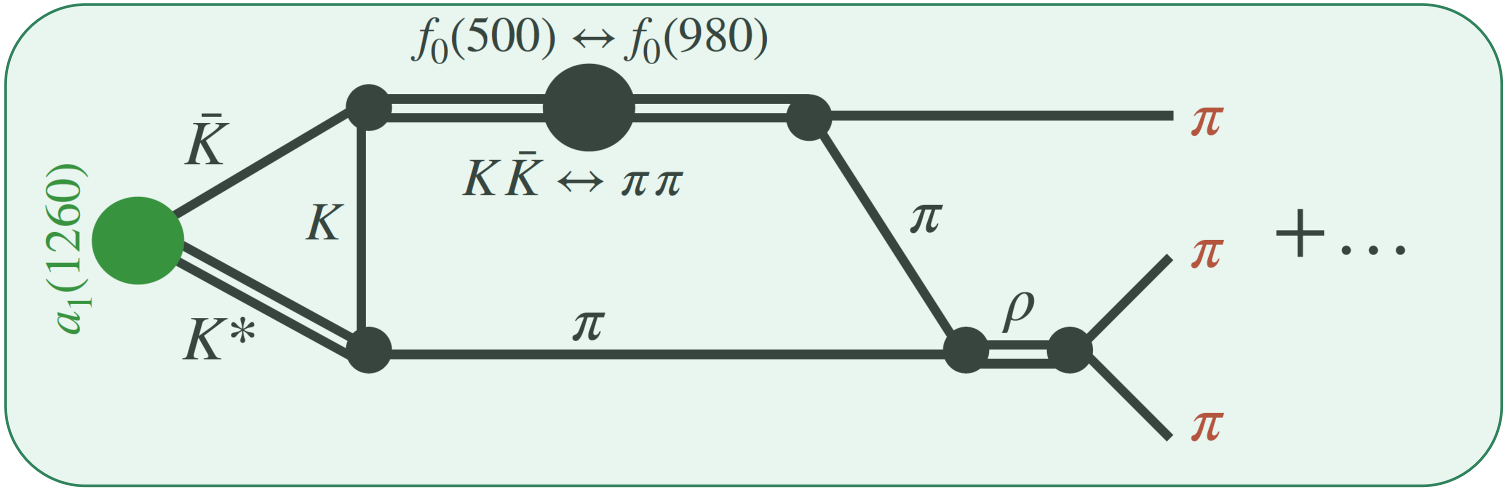
"**Branching ratios**" in 3B decays are momentum -dependent, complex pole residues



Coupled-channel, unitary amplitudes

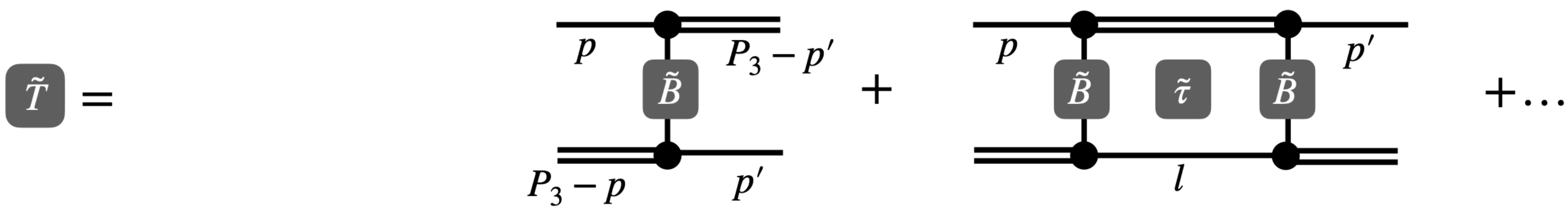
[Feng, Gil, Molina, Mai, Shastry, Szczepaniak, MD, PRD '24]

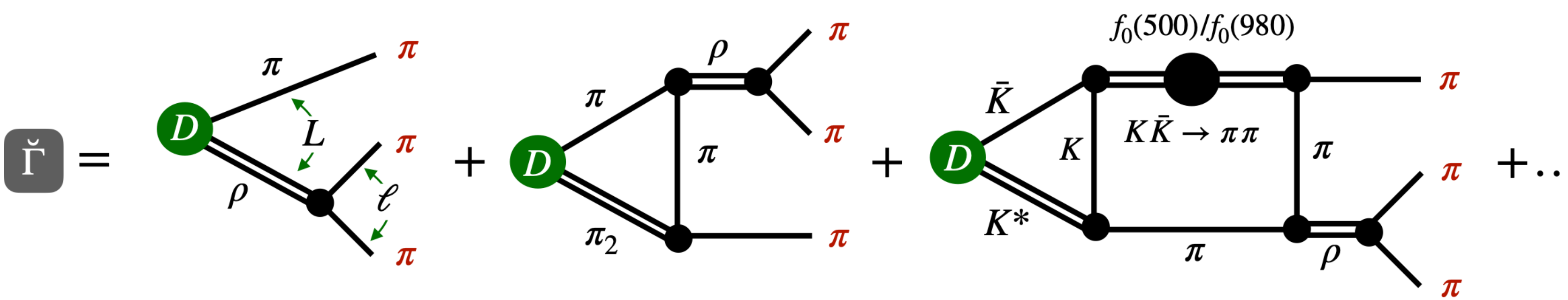
- Coupled-channel, coupled-partial wave amplitudes
- Unitarity manifest
- In-flight transitions of isobars: $\pi\pi \leftrightarrow K\bar{K}$
- All isospins: $I = 0,\ {}^1\!/_2,\ 1, {}^3\!/_2,\ 2$
- All subsystems up to P-wave, including $f_0(500),\ \rho,\ f_0(980), K^*,\ \kappa$
- Example:



Channel space

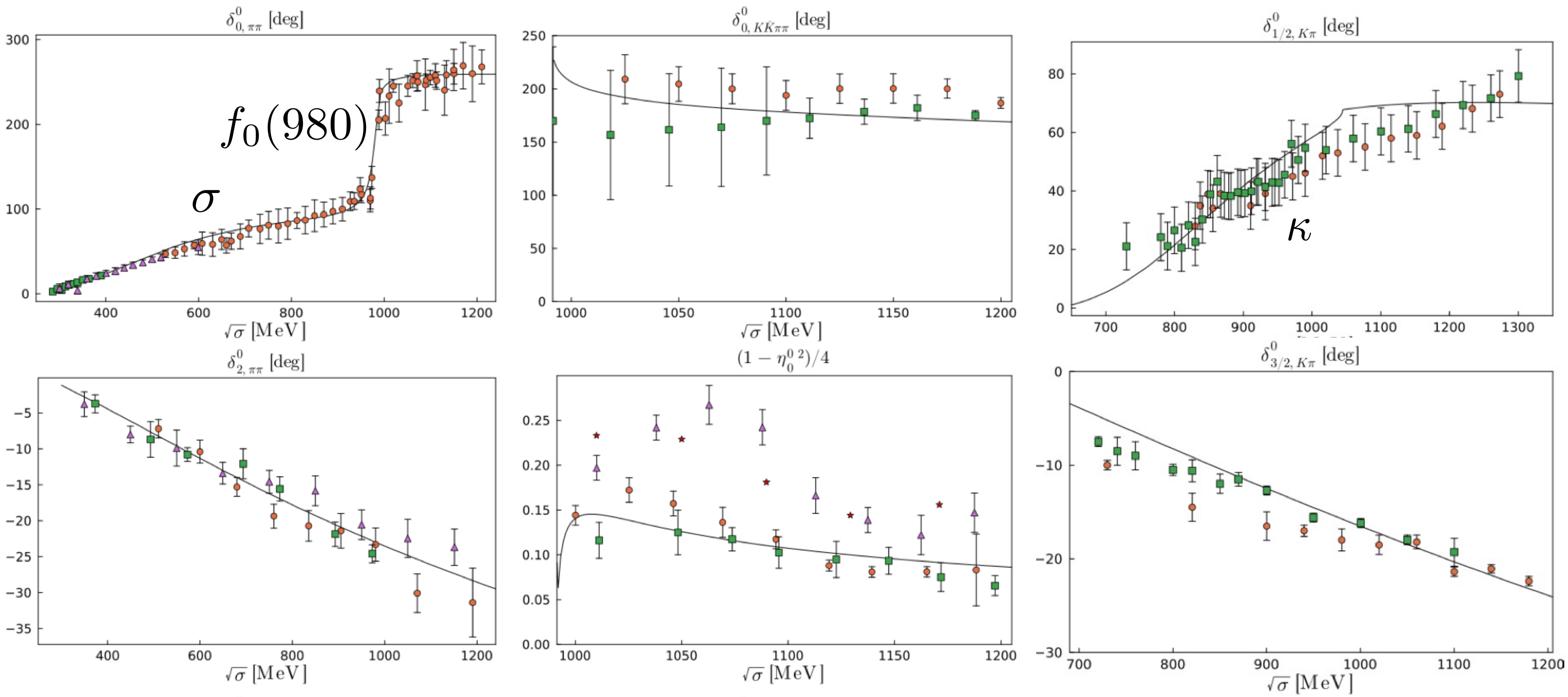
Isobar (S_I, I_I)	$(1,1)$		$(1, 1/2)$		$(0,0)$		$(0,2)$	$(0, 1/2)$	$(0, 3/2)$
HB basis (11 Ch.)	$\pi\rho_{\lambda=\pm1,0}$		$KK^*_{\lambda=\pm1,0}$		$\pi\sigma$	$\pi(K\bar{K})_S$	$\pi\pi_2$	$K\kappa$	$K(\pi K)_S$
JLS basis (9 Ch.)	$(\pi\rho)_S$	$(\pi\rho)_D$	$(KK^*)_S$	$(KK^*)_D$	$(\pi\sigma)_P$	$(\pi(K\bar{K})_S)_P$	$(\pi\pi_2)_S$	$(K\kappa)_S$	$(K(\pi K)_S)_P$

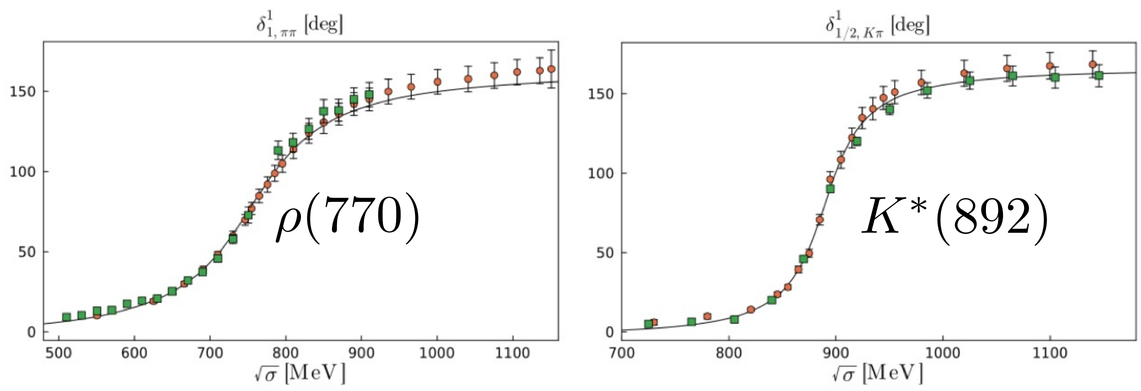




- Scattering matrix dimensions:
  Spectator momentum $\otimes$ JLS channels $\otimes$ isobar channels

Two-body input





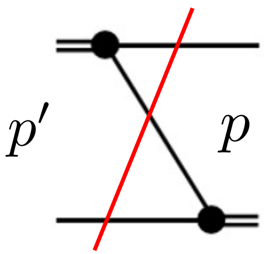
Notation:

$$\delta^{\ell}_{I,i},\ \ i \in \{\pi\pi, K\bar{K}, K\pi\}$$

How to solve the scattering equation

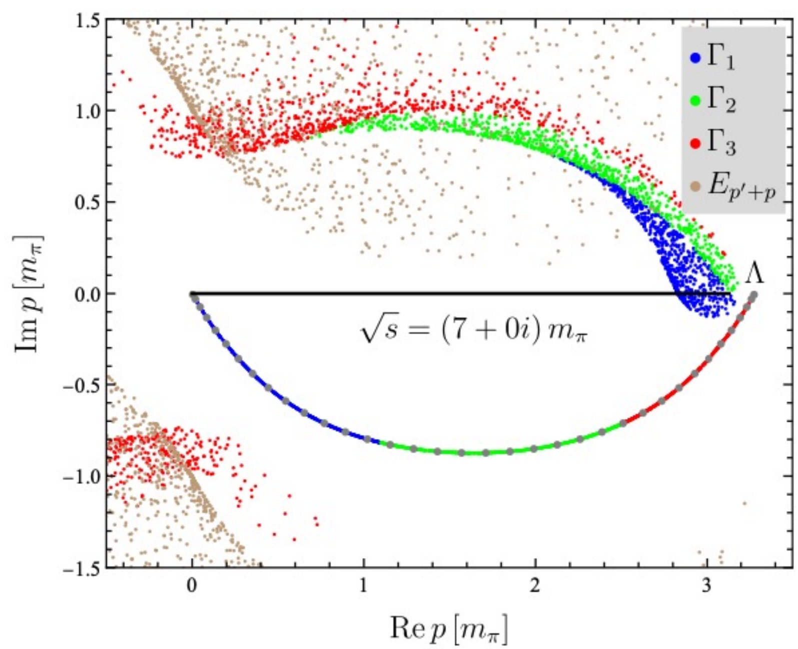
$$\tilde{T}_{ji}(s,p',p) = \tilde{B}_{ji}(s,p',p) + \tilde{C}_{ji}(s,p',p) + \int_0^\Lambda \frac{\mathrm{d}l\, l^2}{(2\pi)^3\, 2E_l} \left(\tilde{B}_{jk}(s,p',l) + \tilde{C}_{jk}(s,p',l)\right) \tilde{\tau}_k(\sigma_l)\, \tilde{T}_{kj}(s,l,p)$$

- **Three-body cuts**



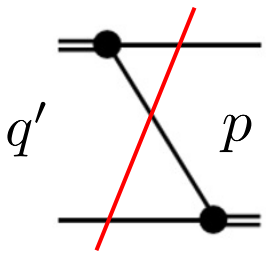
$$\tilde{B}_{ji}(s,\boldsymbol{p}',\boldsymbol{p}) = \frac{(\tilde{\boldsymbol{I}}_F)_{ji}\, v_j^*(p, P-p-p')v_i(p', P-p-p')}{2E_{p'+p}(\sqrt{s} - E_p - E_{p'} - E_{p'+p}) + i\epsilon}$$

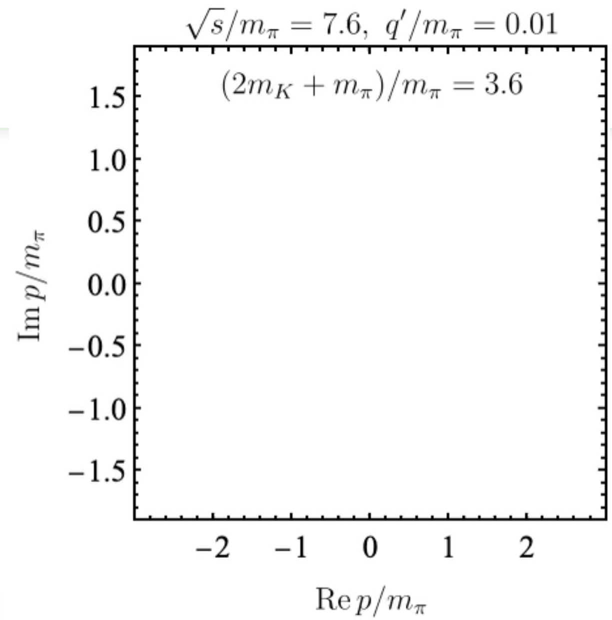
- Angle, energy dependent
- Depend also on *p′* and *p*
- Solve LSE for complex momenta on a deformed contour

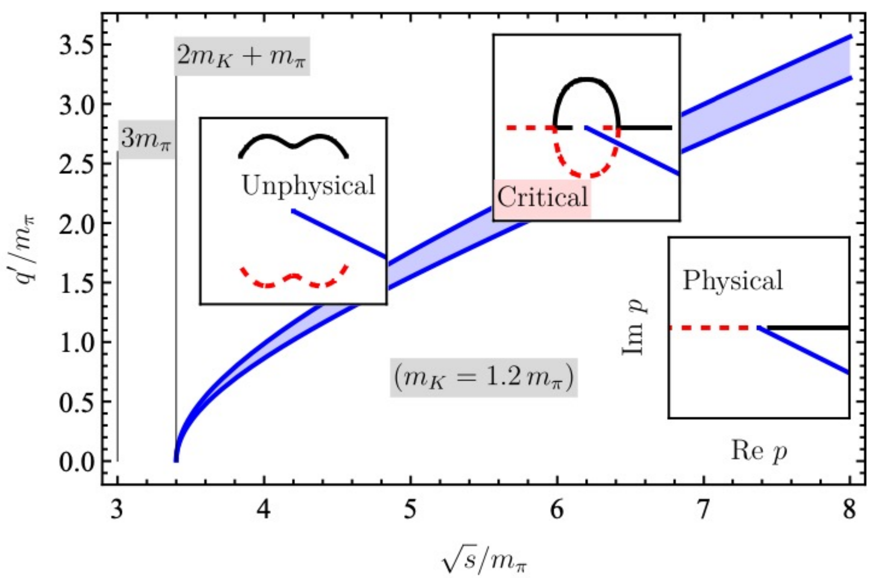


How to get the solution for real, physical momenta

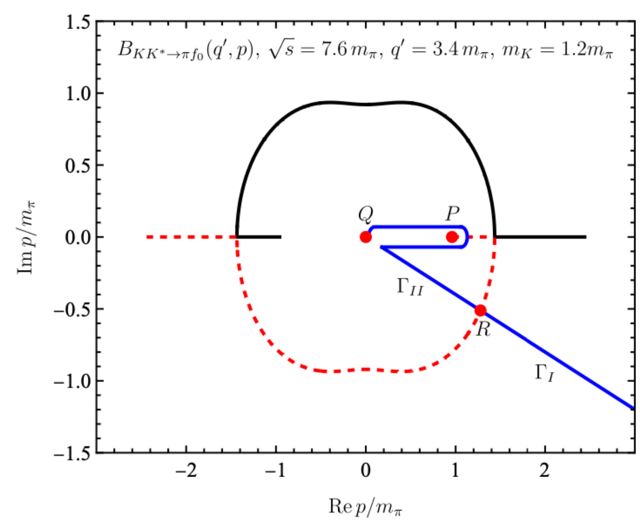
- No solution for real momenta in"critical region"







Solution 1: Contour deformation [Cahill & Sloane]



How to get the solution for real, physical momenta

Solution 2: Direct inversion [Ziegelmann et al.]

Production amplitude

$$\tilde{\Gamma}_j^T(s,q') = D_j(s,q') + \int_0^\Lambda \frac{dq\,q^2}{(2\pi)^3 2E_q}\,\tilde{B}ji(s,q',q)\,\tilde{\tau}_i(\sigma(q))\,\tilde{\Gamma}_i^T(s,q)$$

Ansatz

$$\tilde{\Gamma}^T(q) \approx \sum_{i=1}^{N} \tilde{\Gamma}^T(q_i)\,H_i(q)$$

with Lagrange polynomials

$$H_i(q) = \frac{\prod_{j\neq i}^N (q-q_j)}{\prod_{j\neq i}^N (q_i-q_j)}$$

Makes integral equation a matrix equation

$$\tilde{\Gamma}^T(q_j) = D(q_j) + A_{ji}\tilde{\Gamma}^T(q_i)$$

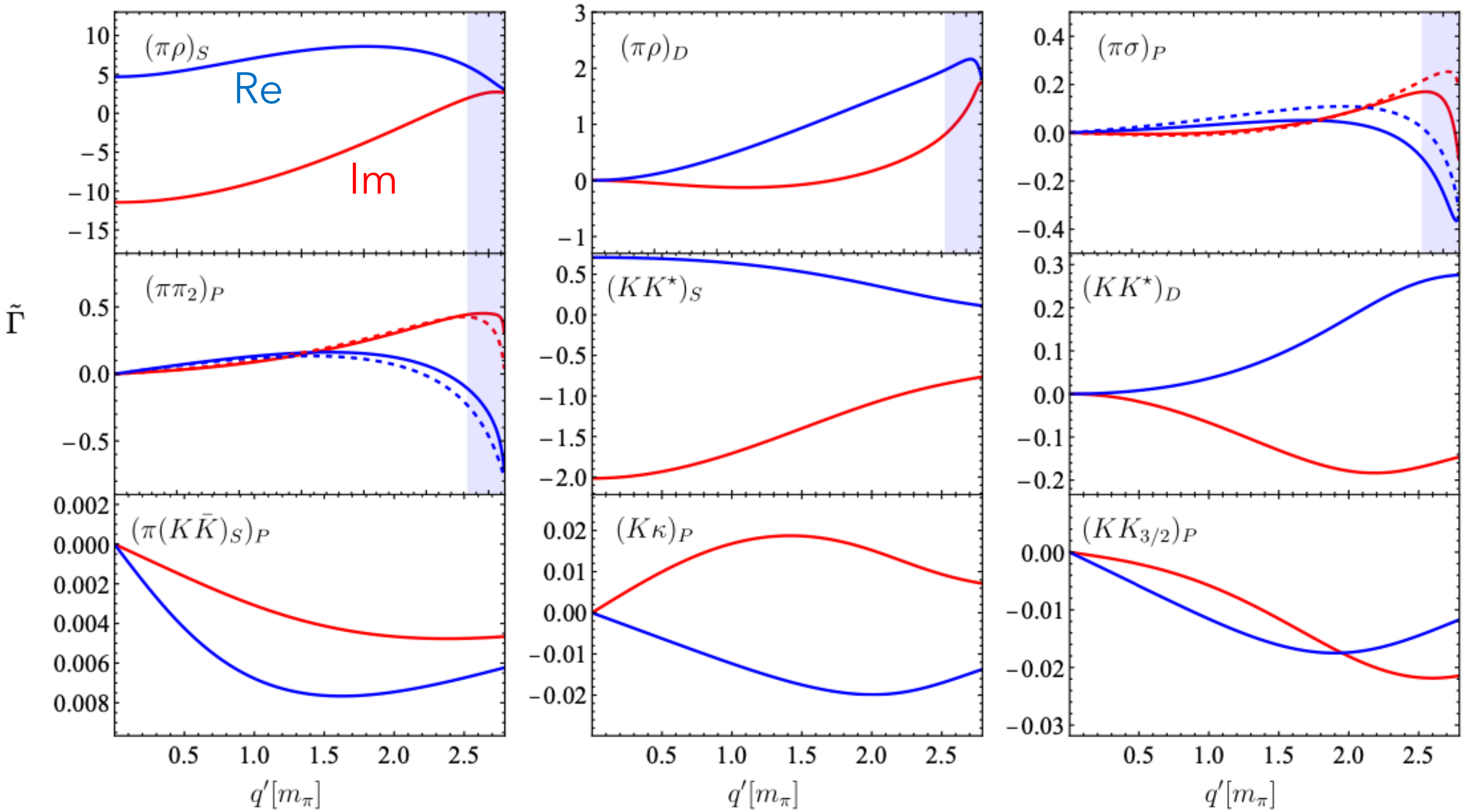
With singular integrals

$$A_{ji} = \int_0^\Lambda \frac{dq\,q^2}{(2\pi)^3 2E_q}\,\tilde{B}(q_j,q)\,\tilde{\tau}(\sigma(q))\,H_i(q)$$

… for which many established algorithms exist

Production amplitude 9-channel model

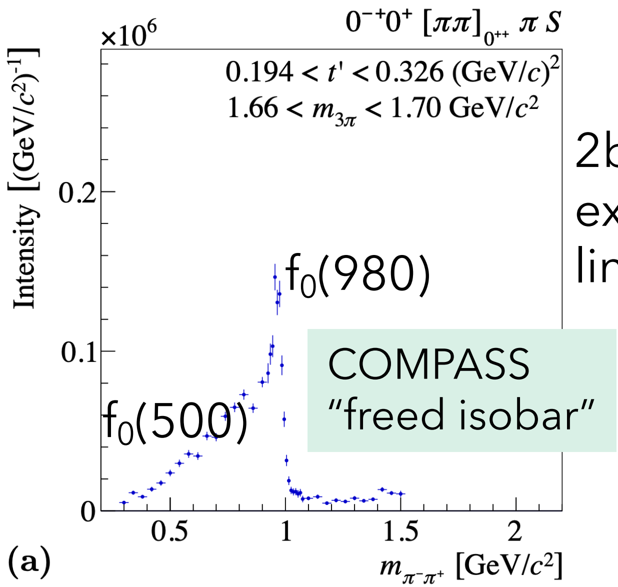
(Only the (non-trivial) rescattering piece) $3m_\pi < W < 2m_K + m_\pi$



Dashed lines: with **πρ** switched off (influence of coupled channels)

Future applications: Line-shape modifications

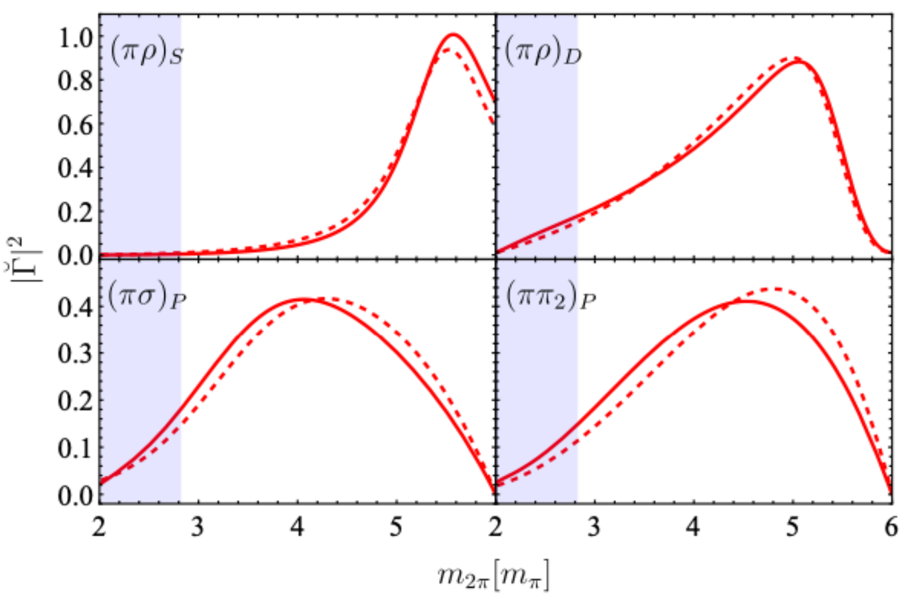
Lineshapes in the analysis of experimental data (COMPASS)



[Phys. Rev. D 95, 032004 (2017)]

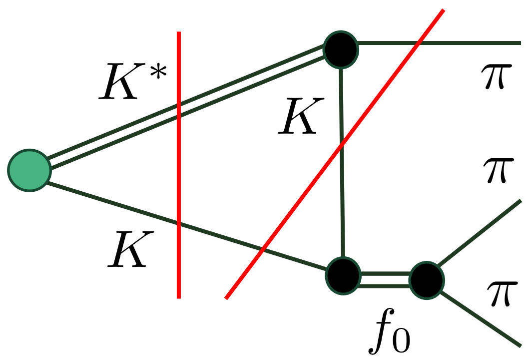
2b lineshape extracted from 3b lineshape

Predicted line-shape modifications by three-body corrections and coupled-channels:



Triangle singularities (TS)

- Triangle singularities are three-body singularities happening in the physical region, while isobar & spectator are also "on-shell"



Notation: K can also stand for $\bar{K}$

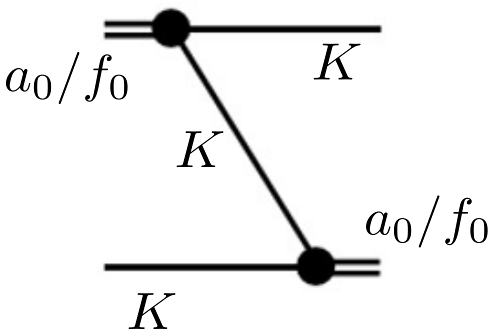
- How does re-scattering affect triangle singularities?
  - Rather small effects [Sakthivasan, Mai, Rusetsky, M.D., arXiV: 2407.17696]

- Quantitative results under way: Unified description of *a_1(1260)* and *a_1(1420)* as resonance + TS

Triangle singularities at thresholds (1)

[Khemchandani, Martinez, MD, in progress]

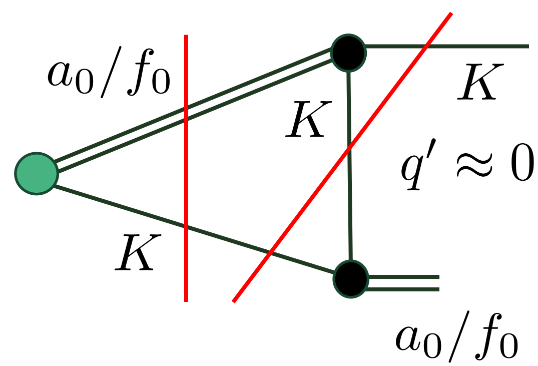
- The K(1460) resonance from Kf_0(980) and Ka_0(980) channels



- Molecular state? [Albaldejo et al., 2010; Martinez et al., 2011]
- At zero a_0, f_0 widths, "molecular "bound state with E_B=0.5MeV is found "dynamically generated" [Zang, Hanhart et al., EPJA 2022]

Preliminary findings: E_B=0.5MeV confirmed. Once isobars have width, an interplay of real thresholds, complex thresholds, triangle singularity, and a molecular state arise.

$$m_{a_0} \approx m_{f_0} \approx 2m_K$$

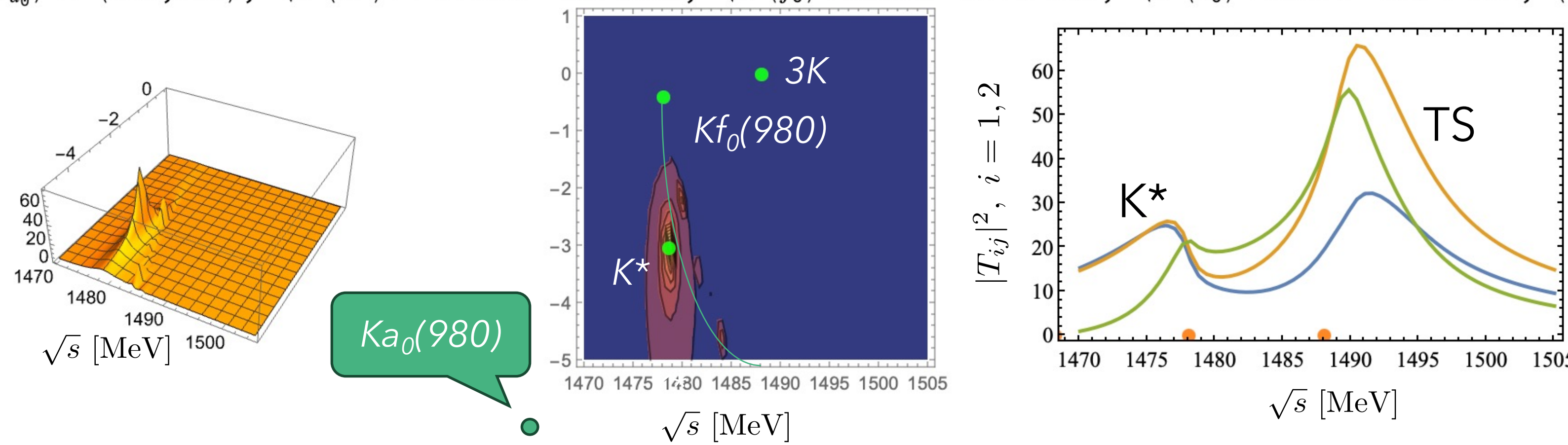


Triangle singularities at thresholds (2)

[Khemchandani, Martinez, MD, prelim]

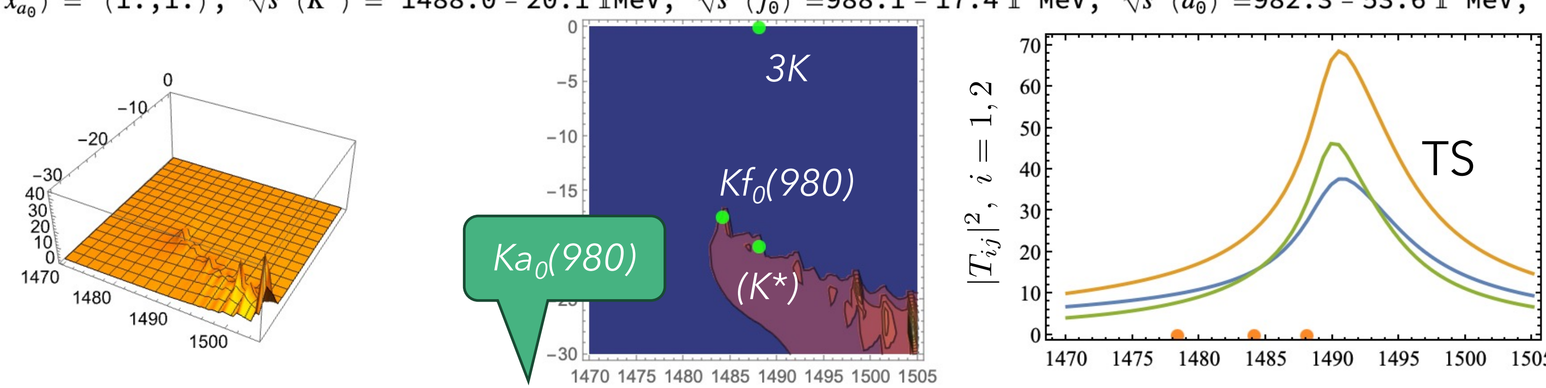
- Isobars with almost zero width

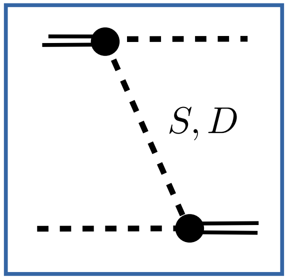
$(x_{f_0}, x_{a_0}) = (0.2,0.2)$, $\sqrt{s}\ (K^*) = 1479.0 - 3.0\ \mathrm{i}$ MeV; $\sqrt{s}\ (f_0) = 982.0 - 0.4\ \mathrm{i}$ MeV; $\sqrt{s}\ (a_0) = 972.1 - 8.2\ \mathrm{i}$ MeV; (n



- Isobars with full width: Only the TS survives; K* pole disappears

$(x_{f_0}, x_{a_0}) = (1.,1.)$, $\sqrt{s}\ (K^*) = 1488.0 - 20.1\ \mathrm{i}$ MeV; $\sqrt{s}\ (f_0) = 988.1 - 17.4\ \mathrm{i}$ MeV; $\sqrt{s}\ (a_0) = 982.3 - 53.6\ \mathrm{i}$ MeV; (





Summary

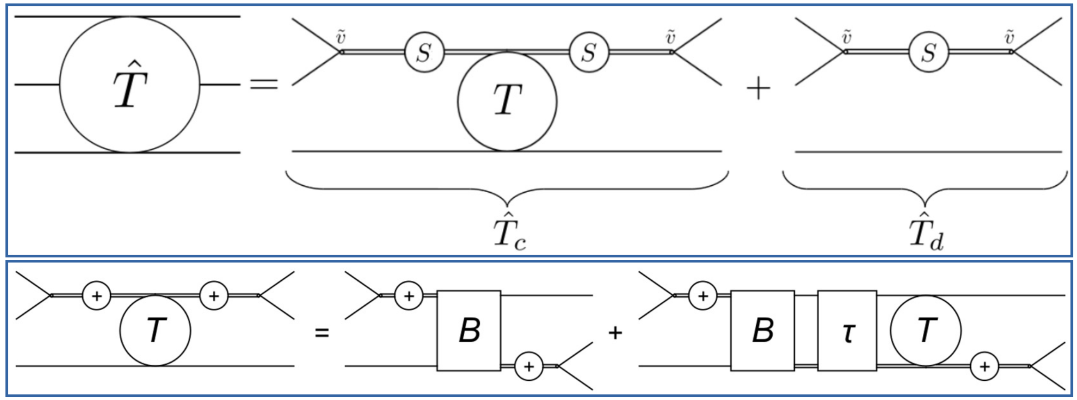
- Lattice QCD progress in determining the explicit dynamics of three-body systems:
  - Three pions at maximal isospin well understood (FVU, RFT, Peng Guo,…)
  - First determination of existence and properties of a three-body resonance – the **a_1(1260)** - in coupled channels by FVU, recently: **ω**
- **Outlook:** More (isospin) channels; other physical systems
  - Channel extension for strangeness with many applications
  - Data analysis (GlueX, Compass, Amber, ?)
  - Dynamical generation of resonances, kinematic effects (triangles), complex branch points: Unified treatment seems possible.
  - Other ideas?

THANK YOU VERY MUCH FOR HOSPITALITY, INTEREST & ATTENTION

Spare slides

Scattering amplitude

$3 \to 3$ scattering amplitude is a 3-dimensional integral equation



LS-type

- Imaginary parts of ***B*, *τ*** $^{-1}$ are fixed by **unitarity/matching**
- B, S are determined **consistently** through 8 different relations

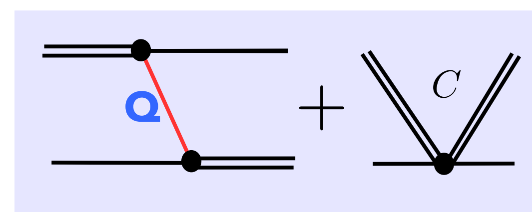
Matching → $\text{Disc}\, B(u) = 2\pi i \lambda^2 \dfrac{\delta\left(E_Q - \sqrt{m^2+\mathbf{Q}^2}\right)}{2\sqrt{m^2+\mathbf{Q}^2}}$

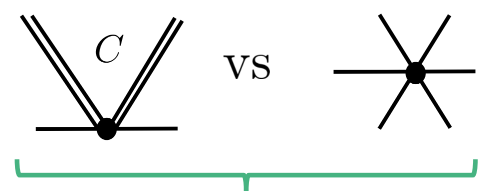
- un-subtracted dispersion relation

$$\langle q|B(s)|p\rangle = -\frac{\lambda^2}{2\sqrt{m^2+\mathbf{Q}^2}\left(E_Q-\sqrt{m^2+\mathbf{Q}^2}+i\epsilon\right)} + C$$

- one-**π** exchange in TOPT → *RESULT, NOT INPUT !*

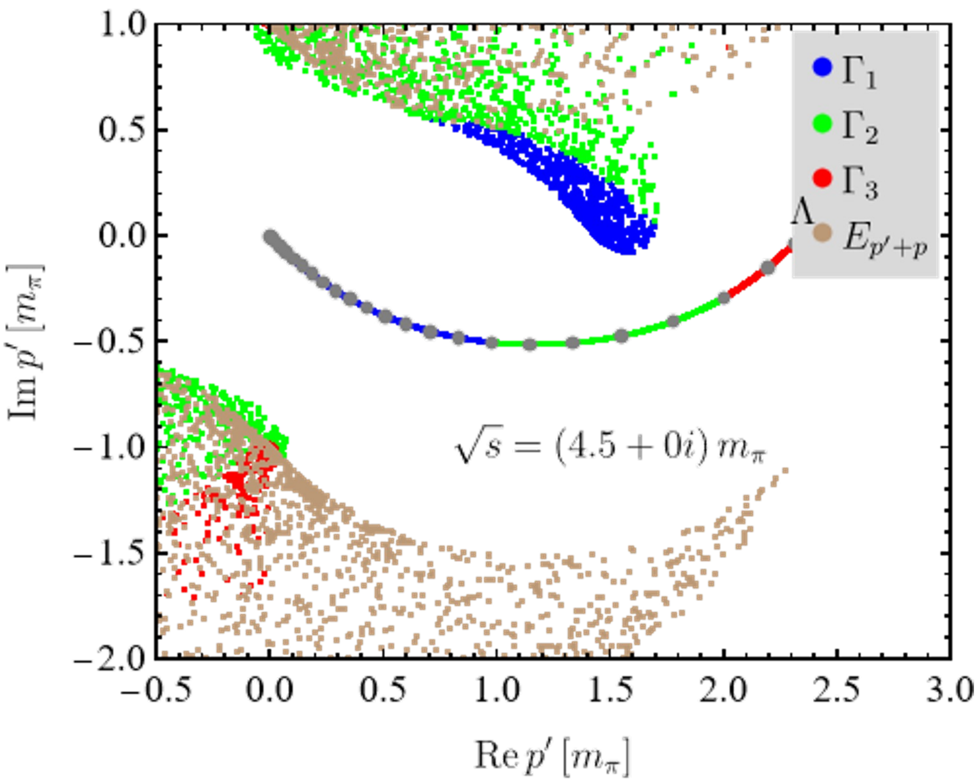
- One <u>can</u> map to field theory but does not have to. Result is a-priori dispersive.





Add. Steps to map to theory might be needed [Brett (2021)]

Details: Solving the scattering equation at complex momenta



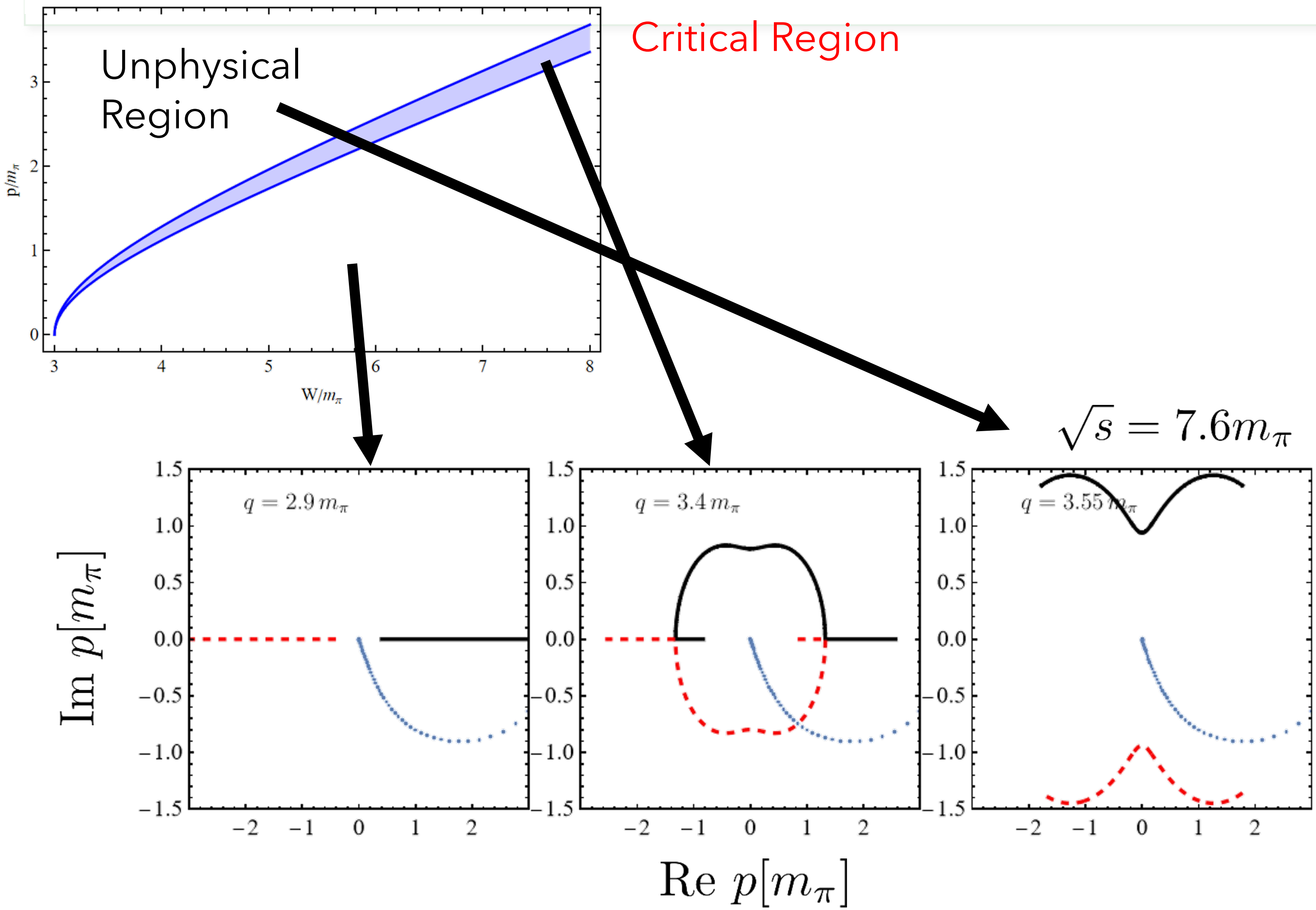
$$f(\boldsymbol{p}', \boldsymbol{p}) = \frac{1}{\sqrt{s} - E_p - E_{p'} - E_{p+p'} + i\epsilon}$$

- Avoid vanishing denominator at

$$p'_{\pm} = \frac{px(p^2 - \alpha^2) \pm \alpha\sqrt{\left(\beta + p^2\,(x^2 - 1)\right)^2 - 4m_\pi^2 \beta}}{2\beta},$$

$$\alpha(p) = \sqrt{s} - E_p\,, \quad \beta(p, x) = \alpha^2(p) - p^2 x^2\,. \quad (23)$$

Spectator momentum regions

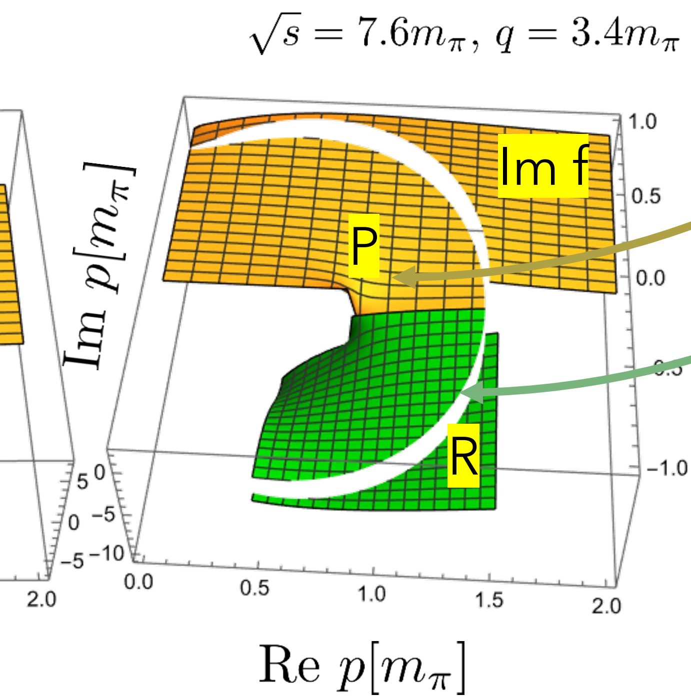


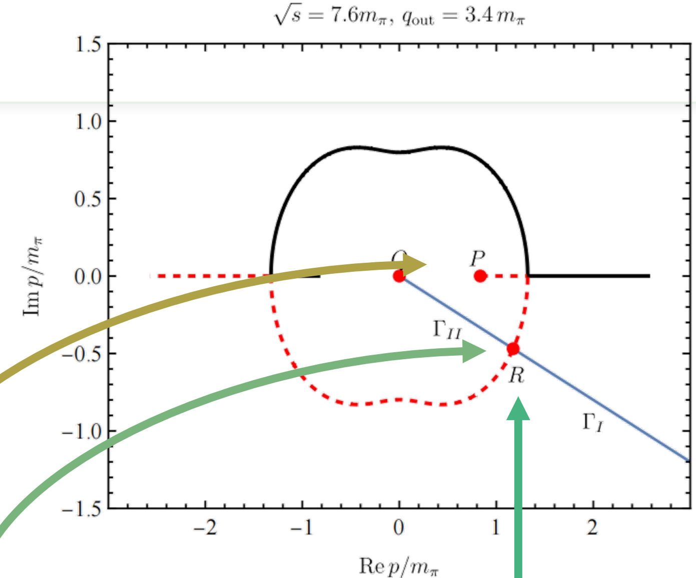
Analytic continuation to 2nd sheet & back

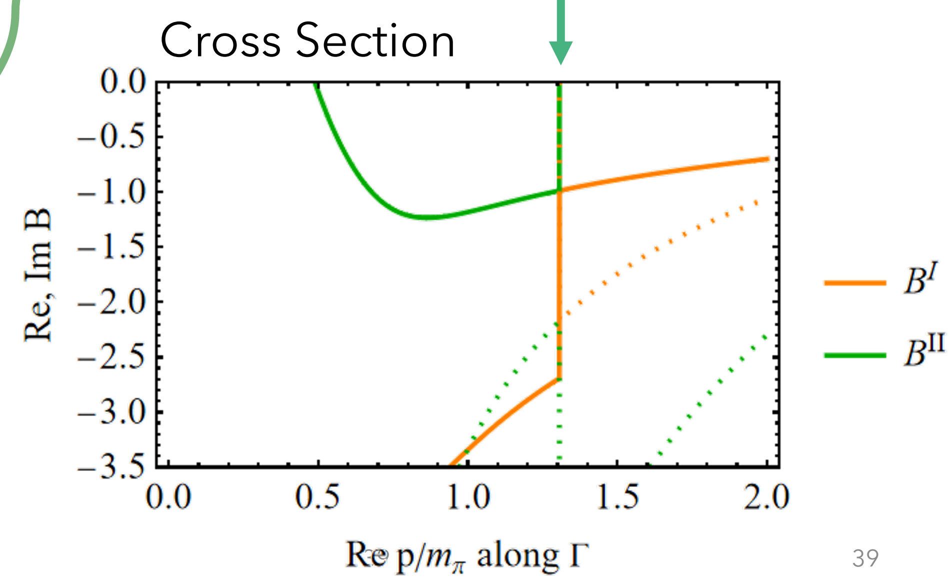
Discontinuity: 2 times

$$f(\boldsymbol{p}', \boldsymbol{p}) = \frac{1}{\sqrt{s} - E_p - E_{p'} - E_{p+p'} + i\epsilon}$$

the smooth transition between first sheet (orange) and second sheet (green).







Partial-wave decomposition

- Plane-wave basis

$$T_{\lambda'\lambda}(\boldsymbol{p},\boldsymbol{q}_1) = (B_{\lambda'\lambda}(\boldsymbol{p},\boldsymbol{q}_1) + C) + \sum_{\lambda''}\int \frac{d^3\boldsymbol{l}}{(2\pi)^3 2E_l}\,(B_{\lambda'\lambda''}(\boldsymbol{p},\boldsymbol{l}) + C)\,\tau(\sigma(l))T_{\lambda''\lambda}(\boldsymbol{l},\mathbf{q}_1)$$

$$B^J_{\lambda\lambda'}(q_1,p) = 2\pi \int_{-1}^{+1} \mathrm{d}x\, d^J_{\lambda\lambda'}(x) B_{\lambda\lambda'}(\boldsymbol{q}_1,\boldsymbol{p}) \qquad B^J_{LL'}(q_1,p) = U_{L\lambda}B^J_{\lambda\lambda'}(q_1,p)U_{\lambda'L'}$$

- JLS basis:

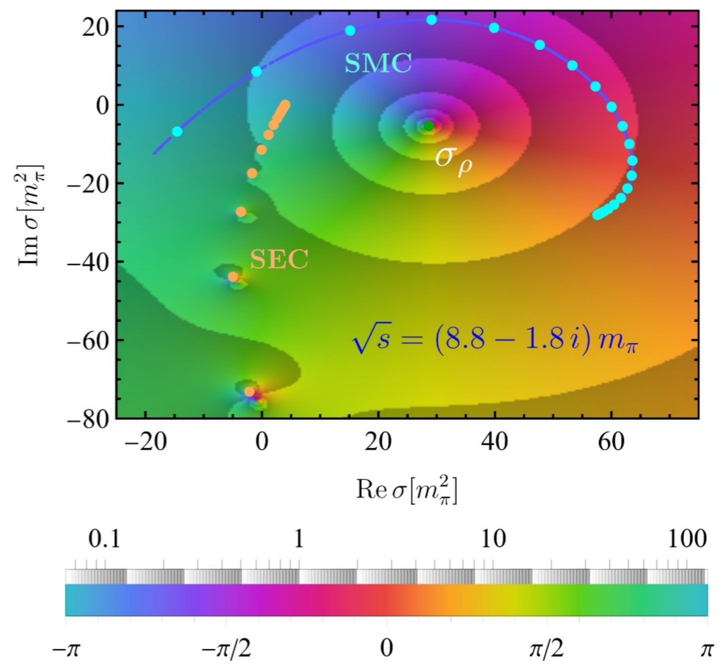
$$T^J_{LL'}(q_1,p) = \left(B^J_{LL'}(q_1,p) + C_{LL'}(q_1,p)\right) + \int_0^\Lambda \frac{\mathrm{d}l\, l^2}{(2\pi)^3 2E_l}\left(B^J_{LL''}(q_1,l) + C_{LL''}(q_1,l)\right)\tau(\sigma(l))T^J_{L''L'}(l,p)$$

Analytic cont. 3-body

[Sadasivan (2021)]
[Doering (2009)]

SMC

$$T^J_{LL'}(q_1,p) = \left(B^J_{LL'}(q_1,p) + C_{LL'}(q_1,p)\right) + \int_0^\Lambda \frac{dl\, l^2}{(2\pi)^3 2E_l} \left(B^J_{LL''}(q_1,l) + C_{LL''}(q_1,l)\right)\tau(\sigma(l))T^J_{L''L'}(l,p)$$

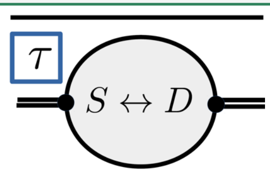


$$\tau^{-1}(\sigma) = K^{-1} - \Sigma\,,$$

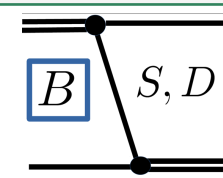
$$\Sigma = \int_0^\infty \frac{dk\, k^2}{(2\pi)^3} \frac{1}{2E_k} \frac{\sigma^2}{\sigma'^2} \frac{\tilde{v}(k)^*\tilde{v}(k)}{\sigma - 4E_k^2 + i\epsilon}$$

$$B_{\lambda\lambda'}(\boldsymbol{p},\boldsymbol{p}') = \frac{v^*_\lambda(P-p-p',p)\,v_{\lambda'}(P-p-p',p')}{2E_{p'+p}\left(\sqrt{s} - E_p - E_{p'} - E_{p'+p} + i\epsilon\right)}$$

SEC



Singularities

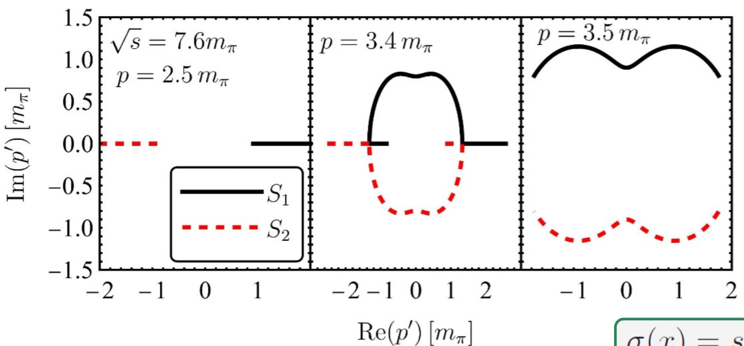


- Two contours (SMC and SEC)
- Deform both "adiabatically" to go to complex s
- Set of rules:
  - Contours cannot intersect with each others
  - Contours cannot intersect with (3-body) cuts
- Passing singularities left or right determines sheet

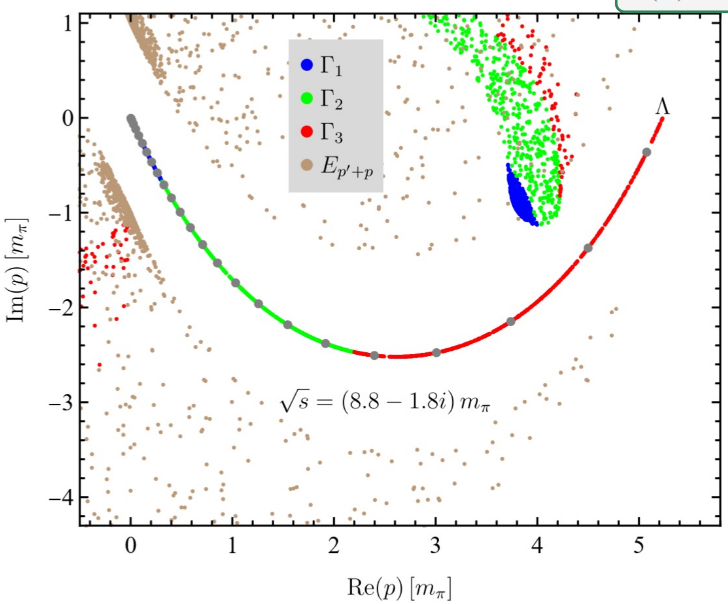
Analytic continuation 3-body (contd.)

- Three-body cuts

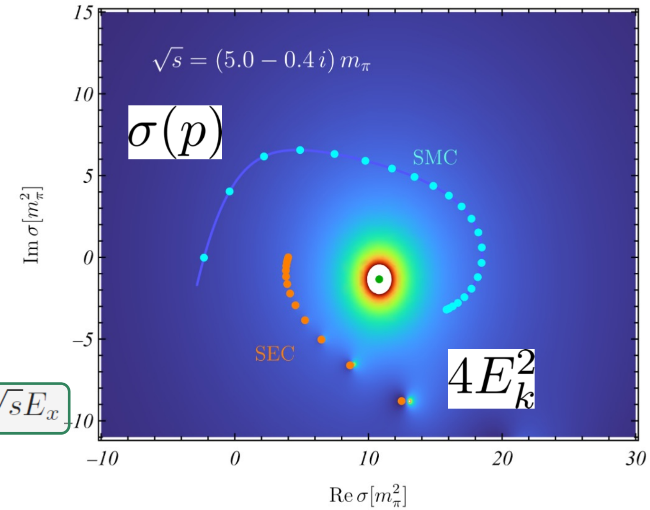
$$\sqrt{s} - E_p - E_{p'} - E_{p+p'} + i\epsilon = 0$$



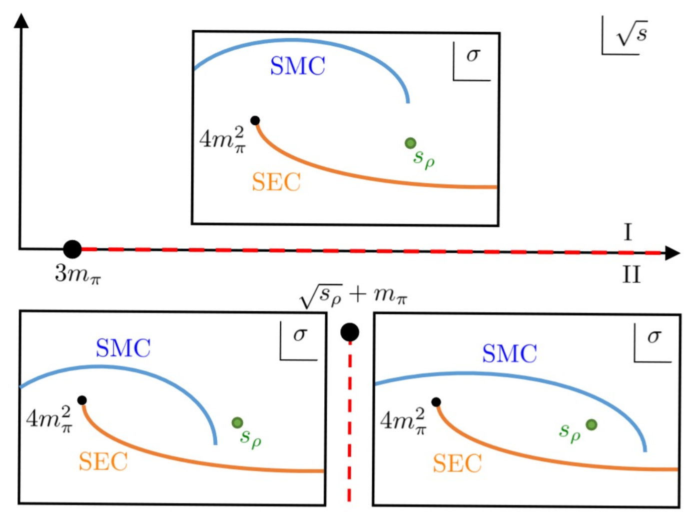
$$\sigma(x) = s + m_\pi^2 - 2\sqrt{s}E_x$$



- Complex branch points

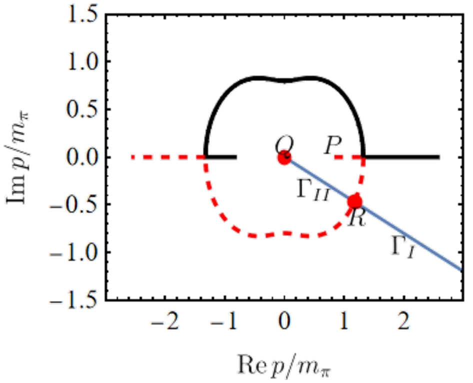


Integration limits at poles induce branch points



Heatherington and Schick method

$$T(s,q,p) = B^I(s,q,p) - \int_0^{P(q)} \frac{dq''\,(q'')^2}{(2\pi)^3 2E_{q''}} \,\mathrm{Im}\, B^{II}(s,q,q'')\,\tau(\sigma(q''))T(s,q'',p)$$
$$+ \int_{\Gamma_{II}} \frac{dl\, l^2}{(2\pi)^3\, 2E_l} B^{II}(s,q,l)\tau(\sigma(l))T(s,l,p) + \int_{\Gamma_I} \frac{dl\, l^2}{(2\pi)^3\, 2E_l} B^{I}(s,q,l)\tau(\sigma(l))T(s,l,p)\;,$$



$\mathrm{Im}\ B^I = 0\,(0 < q'' < P)$ $\quad$ $\mathrm{Im}\ B^{II} \to 2d(s,q,q'')$

$$\tilde\Gamma_{L'}(s,q) = -2\int_0^{P(q)} \frac{dq''\,(q'')^2}{(2\pi)^3 2E_{q''}}\, d_{L'L}(s,q,q'')\,\tau_L(\sigma(q''))\left(\tilde\Gamma_L(s,q'') + D_L(s,q'')\right)$$
$$+ \int_\Gamma \frac{dl\, l^2}{(2\pi)^3\, 2E_l} \left(B^{II\to I}_{L'L}(s,q,l) + C_{L'L}(s,q,l)\right)\tau_L(\sigma(l))\left(\tilde\Gamma_L(s,l) + D_L(s,l)\right)$$

B defined as smooth transition from 2^{nd} to 1^{st} sheet for more compact notation

Production amplitude

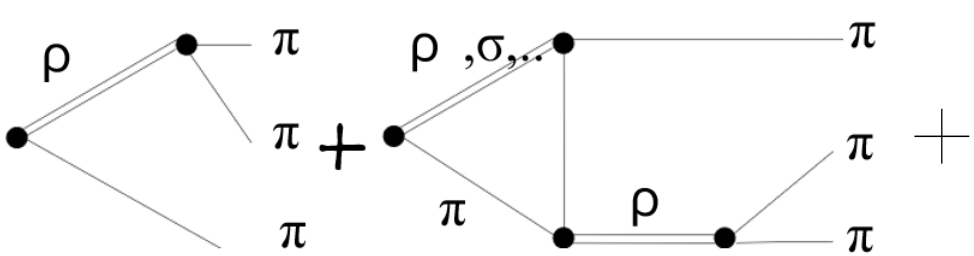
$$\breve{\Gamma} = \breve{\Gamma}^C + \breve{\Gamma}^D$$

$$\breve{\Gamma}_{L'}(s,q) = \breve{v}_{L'}(q)\tau_{L'}(\sigma(q)) \left[\tilde{\Gamma}_{L'}(s,q) + D_{L'}(s,q)\right]$$

2-body decay vertex,
in the 2-body cm
frame

$$D_L(s,p) = D_{fL}(s,p)B_L(\lambda p)$$

Fit to
data



….. (more re-scattering terms
+ 3-body forces)

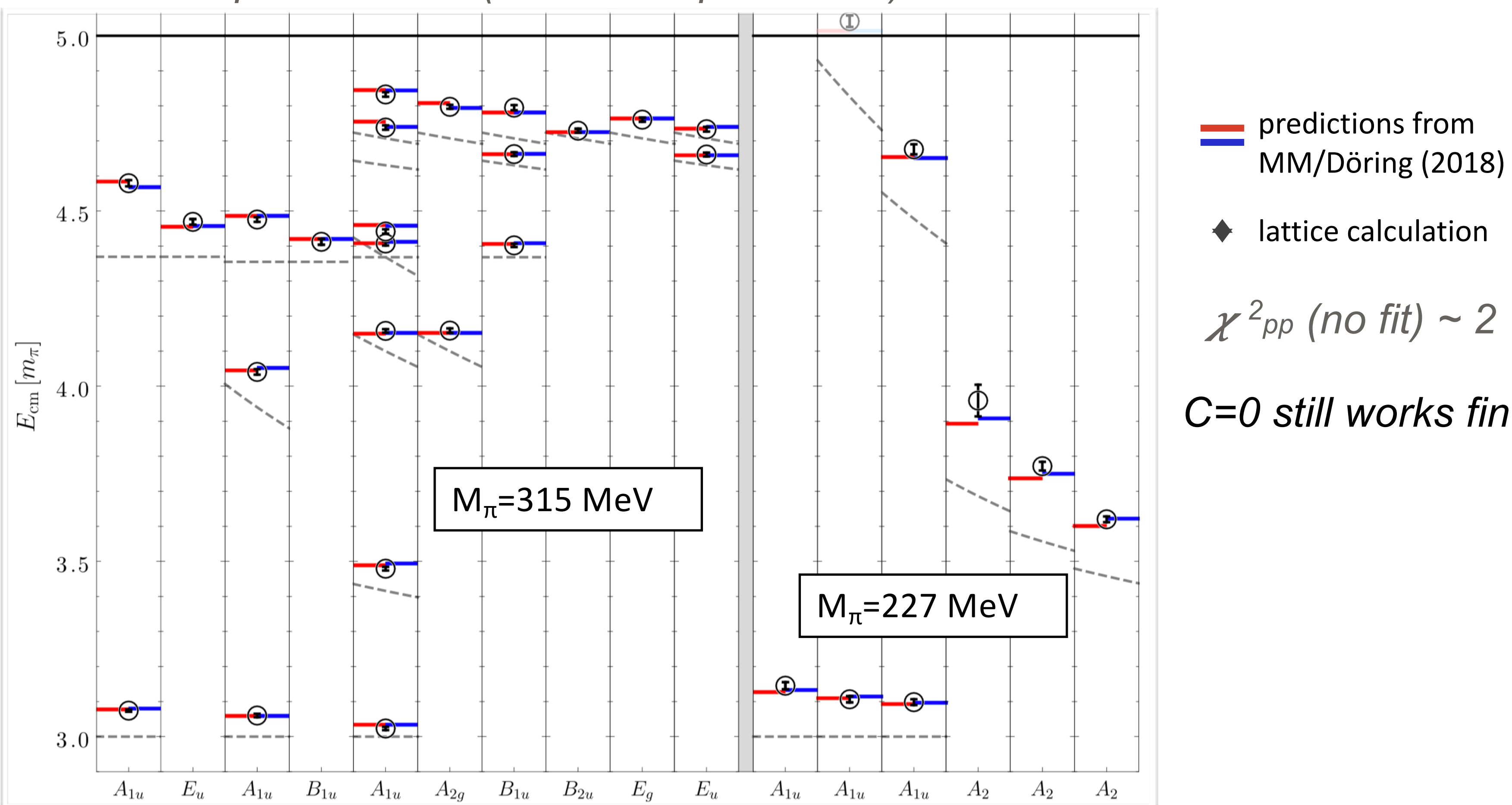
Disconnected

Connected

GWUQCD data

Culver, MM, Brett, Alexandru, Döring (2019) PRD

- ***More recent*** *data is available*
  - *very dense spectrum from elongated boxes*
  - *different pion masses (chiral extrapolations?)*



χ^2_{pp} (no fit) ~ 2

C=0 still works fine

Plane-wave implementation of the C-term

- **Step 1**: JM-basis → Helicity basis
- **Step 2**: partial-wave basis → Plane-wave basis
- **Step 3**: C (and B, and 3B propagator) from plane-wave basis to irreps by suitable rotations

$$\mathcal{A}_{\lambda'\lambda}(s,\boldsymbol{p}',\boldsymbol{p}) = \sum_{M=-J}^{J} \frac{2J+1}{4\pi}\,\mathfrak{D}^{J*}_{M\lambda'}(\phi_{\boldsymbol{p}'},\theta_{\boldsymbol{p}'},0)\,\mathcal{A}^{J}_{\lambda'\lambda}(s,p',p)\,\mathfrak{D}^{J}_{M\lambda}(\phi_{\boldsymbol{p}},\theta_{\boldsymbol{p}},0)\,,$$

Step 2

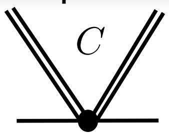
$$\mathcal{A}^{J}_{\lambda'\lambda}(s,p',p) = U_{\lambda'\ell'}\mathcal{A}_{\ell'\ell}(s,p',p)U_{\ell\lambda}\,,$$

$$U_{\ell\lambda} := \sqrt{\frac{2\ell+1}{2J+1}}(\ell 0 1\lambda|J\lambda)(1\lambda 00|1\lambda)) = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}, \\ \frac{1}{\sqrt{6}} & -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{6}}, \end{pmatrix}\,,$$

Step 1

4 different fits to 2 energy eigenvalues

- Fitted isobar-spectator interaction (case 1, 2) for $|\boldsymbol{p}| \leq 2\pi/L|(1,1,0)| \approx 2.69\ m_\pi$



$$C_{\ell'\ell}(s,\boldsymbol{p}',\boldsymbol{p}) = \sum_{i=-1}^{\infty} c^{(i)}_{\ell'\ell}(\boldsymbol{p}',\boldsymbol{p})(s-m_{a_1}^2)^i$$

- a_1 can be generated as pole even though no built-in singularity

Non-zero coefficients	No of fit parameters	χ^2
c_{00}^0 (no built-in pole)	1	9
c_{00}^0, c_{00}^1 (no built-in pole)	2	0.15
g_0, g_2, m_{a1}, c	4	10^{-7}

$$C_{\ell'\ell}(s,\boldsymbol{p}',\boldsymbol{p}) = g_{\ell'}\left(\frac{|\boldsymbol{p}'|}{m_\pi}\right)^{\ell'} \frac{m_\pi^2}{s-m_{a_1}^2} g_\ell \left(\frac{|\boldsymbol{p}|}{m_\pi}\right)^{\ell} + c\,\delta_{\ell'0}\delta_{\ell 0}$$

- In these cases, there is a built-in singularity, leading to resonance poles

Three kaons at maximal isospin

[Alexandru 2020]

- First study of three kaons from lattice QCD with chiral amplitudes
- Other groups have improved on this in the meantime:
  - Max. isospin, non-identical masses ($\pi^+\pi^+K^+$, $\pi^+K^+K^+$) [Blanton 2021]
  - Pions and kaons at maximal isospin with unprecedented accuracy and no. of levels ($\pi^+\pi^+\pi^+$, $K^+K^+K^+$) [Blanton 2021]

- Two mass-degenerate light quarks (u,d); valence strange quark
- nHYP-smeared clover action
- quark propagation is treated using the LapH method with optimized inverters
- Lattice spacing determined from Wilson flow parameter w_0