Amplitudes in Helicity Formalism and Polarization Effects,v2

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Abstract

Taking the process $\overline{p}p \to \omega\pi^0, \omega \to \pi^0\gamma$ as example the ρ matrix of the final state is derived in the helicity formalism taking into account P- and C-conservation.The differential cross section and the gamma polarization observables are calculated using Mathematica.A comparison with similar calculations for the decay $\omega \to \pi^+\pi^-\pi^0$ is performed.

1 Cross sections and polarization observables

Wave functions of scattering processes $(a + b \to c + d)$:
$\psi = e^{\imath kz} + \frac{e^{k'r}}{r} f(\Theta, \Phi)$ with f=scattering amplitude
The scattering amplitude has three independent parts:
Momentum space(see above)
Spin space(helicity description,e.g.):$f_{\lambda_c,\lambda_d,\lambda_a,\lambda_b}(\Theta, \Phi)$
Isospin space
More details can be found in [1, 3, 2, 4, 5, 6, 7, 8, 9].

Example for the spin part of the amplitude:$\overline{p}p \to \omega\pi^0, \omega \to \gamma\pi^0$
$f_{\lambda_\gamma,0,\lambda_{\overline{p}},\lambda_p}(\Theta, \Phi, \theta, \phi) = \frac{1}{2p} \sum_J \sum_{\lambda_\omega} (2J+1) \sqrt{\frac{2\times1+1}{4\pi}} D^{1*}_{\lambda_\omega \lambda_\gamma}(\phi, \theta) \times$

$\times \langle 0, \lambda_\gamma \left| T^1 \right| 1^- \rangle \langle 0, \lambda_\omega \left| T^J \right| \lambda_{\overline{p}}, \lambda_p \rangle d^J_{\nu\mu}(\Theta) e^{\imath\nu\Phi}$ (1)
with $\nu = \lambda_{\overline{p}} - \lambda_p; \mu = \lambda_\omega; D = D(\phi, \theta, 0)$;p= CM momentum of $\overline{p}/p; \Phi, \Theta$=production angles of the $\omega; \phi, \theta$ = decay angles of the Gamma.
In the case of an unpolarized initial state(see below) Φ can be set to zero and ϕ is measured relative to the $\omega\pi^0$ production plane(see Panda Physics Note:AN-QCD-2012-001.pdf)

The amplitude is expanded in angular momentum states(J),which have to be added coherently.In the example above the amplitude is a 12-dimensional matrix,with 2-dimensional spinors as eigenstates of the antiproton/proton $(|\lambda_{\overline{p}}, |\lambda_p\rangle)$ and a 3-dimensional vector($\langle\lambda_\gamma|$) for the γ (for the real photon the dimension of the vector can be reduced to 2,resulting in a 8-dimensional scattering matrix).The spin state of the massive omega has three components(λ_ω),but does not appear in the final amplitude because of the coherent summation on it.

Calculation of differential cross sections:
The differential cross section is given by
$d\sigma_\rho/d\Omega = tr(\rho^f) = tr(f\rho^{in}f^\dagger)$.
ρ^{in} and $\rho^f = f\rho^{in}f^\dagger$ are the spin density matrices of the initial and final states,respectively,with tr(ρ^{in})=1 and tr(ρ^f)= $d\sigma_\rho/d\Omega$.

For the example above:$d\sigma_\rho/d\Omega = \sum_{\lambda_\gamma}\sum_{\lambda_{\overline{p}},\lambda_p}\sum_{\lambda_{\overline{p}'},\lambda_{p'}} f_{\lambda_\gamma,0,\lambda_{\overline{p}},\lambda_p}(\Theta,\Phi,\theta,\phi)\rho^{in}_{\lambda_{\overline{p}}\lambda_p;\lambda_{\overline{p}'}\lambda_{p'}}\times$
$\times f^\dagger_{\lambda_\gamma,0,\lambda_{\overline{p}'},\lambda_{p'}}(\Theta,\Phi,\theta,\phi)$ with $d\Omega = dcos\Theta dcos\theta d\Phi d\phi$

Here it is assumed that no gamma polarization is measured,only the momentum of the outgoing gamma.The index ρ indicates,that the expression is valid for arbitrary ρ^{in}.
The expectation values of observables A,represented by the operators(matrices) $\underline{A}$ are given by
$\langle A\rangle = (d\sigma_\rho/d\Omega)^{-1}tr(\underline{A}\rho^f) = (d\sigma_\rho/d\Omega)^{-1}tr(\underline{A}f\rho^{in}f^\dagger)$
Examples for $\underline{A}$ are e.g. the spin components $\underline{S}$.

For the example above:
$\langle A\rangle = \sum_{\lambda_\gamma,\lambda_{\gamma'}} A_{\lambda_{\gamma'},\lambda_\gamma}\rho^f_{\lambda_\gamma,\lambda_{\gamma'}} = \sum_{\lambda_{\gamma'}}(\underline{A}\rho^f)_{\lambda_{\gamma'}\lambda_{\gamma'}} =$
$= (d\sigma_\rho/d\Omega)^{-1}\sum_{\lambda_\gamma,\lambda_{\gamma'}}\sum_{\lambda_{\overline{p}},\lambda_p}\sum_{\lambda_{\overline{p}'},\lambda_{p'}} A_{\lambda_{\gamma'},\lambda_\gamma} f_{\lambda_\gamma,0,\lambda_{\overline{p}},\lambda_p}(\Theta,\Phi,\theta,\phi)\rho^{in}_{\lambda_{\overline{p}}\lambda_p;\lambda_{\overline{p}'}\lambda_{p'}} f^\dagger_{\lambda_{\gamma'},0,\lambda_{\overline{p}'},\lambda_{p'}}(\Theta,\Phi,\theta,\phi)$
(see [2],page 277)
More details can be found in [1](pages 315,..)

There is an important special case:The initial state particles are not polarized,thus an average over the initial spin states has to be performed.This case is assumed throughout the rest of the note.

For the example above:
$\rho^{in} = (2s_{\overline{p}}+1)^{-1}(2s_p+1)^{-1}\times\underline{1} = 1/4\times\underline{1}$ with $\underline{1}$ being the unit matrix.
In components:
$\rho^{in}_{\lambda_{\overline{p}}\lambda_p;\lambda_{\overline{p}'}\lambda_{p'}} = (2s_{\overline{p}}+1)^{-1}(2s_p+1)^{-1}\delta_{\lambda_{\overline{p}}\lambda_{\overline{p}'}}\delta_{\lambda_p\lambda_{p'}} = 1/4\delta_{\lambda_{\overline{p}}\lambda_{\overline{p}'}}\delta_{\lambda_p\lambda_{p'}}$

$\rho^f_{\lambda_\gamma;\lambda_{\gamma'}} = (d\sigma/d\Omega)^{-1} \sum_{\lambda_{\overline{p}},\lambda_p} f_{\lambda_\gamma,0,\lambda_{\overline{p}},\lambda_p}(\Theta,\theta,\phi) f^\dagger_{\lambda_{\gamma'},0,\lambda_{\overline{p}},\lambda_p}(\Theta,\theta,\phi)$
and
$d\sigma/d\Omega = tr(\rho^f) = tr(f\rho^{in}f^\dagger) = (2s_{\overline{p}}+1)^{-1}(2s_p+1)^{-1} \sum_{\lambda_\gamma} \sum_{\lambda_{\overline{p}},\lambda_p} f_{\lambda_\gamma,0,\lambda_{\overline{p}},\lambda_p}(\Theta,\theta,\phi)\times$
$\times f^+_{\lambda_\gamma,0,\lambda_{\overline{p}},\lambda_p}(\Theta,\theta,\phi) =$
$= 1/4 \sum_{\lambda_\gamma} \sum_{\lambda_{\overline{p}},\lambda_p} |f_{\lambda_\gamma,0,\lambda_{\overline{p}},\lambda_p}(\Theta,\theta,\phi)|^2$
with $d\Omega = dcos\Theta dcos\theta d\phi$(for unpolarized initial states no dependence on Φ appears)
and
$\langle A\rangle = tr(\rho^f \underline{A}) = (d\sigma/d\Omega)^{-1} \sum_{\lambda_\gamma,\lambda_{\gamma'}} \sum_{\lambda_{\overline{p}},\lambda_p} A_{\lambda_{\gamma'},\lambda_\gamma} f_{\lambda_\gamma,0,\lambda_{\overline{p}},\lambda_p}(\Theta,\theta,\phi) f^\dagger_{\lambda_{\gamma'},0,\lambda_{\overline{p}},\lambda_p}(\Theta,\theta,\phi)$

⌈ Remark on pure/mixed states:
The amplitude given in (1) originates from a pure initial state.Thus the gamma in the final state is also in a pure state.When an average over the initial states is performed-as it is assumed in the following-the initial state is a mixed state,so is the final gamma state.⌋

2 Helicity Amplitude with P-,C-,T-conservation

These aspects are discussed in detail in [1, 4, 5, 6, 7, 8].Here only the results are given for two-particle states with given J.
Effect of P/C/T operations on two particle helicity states(given J):
P $|JM\lambda_1\lambda_2\rangle = \eta_1\eta_2(-1)^{J-S_1-S_2}|JM-\lambda_1-\lambda_2\rangle$
C $|JM\lambda_1\lambda_2\rangle = (-1)^J|JM\lambda_2\lambda_1\rangle$(Particle/Antiparticle State)
C $|JM\lambda_1\lambda_2\rangle = C_1C_2|JM\lambda_2\lambda_1\rangle$(Particles are eigen states of C)
O(T) $|JM\lambda_1\lambda_2\rangle = (-1)^{J-M}|J-M\lambda_1\lambda_2\rangle$(O(T)=anti-unitary operator)

Effect on reaction amplitudes:1+2→3+4
P-Conservation:
$\langle\lambda_3\lambda_4|T^J|\lambda_1\lambda_2\rangle = \eta_1\eta_2\eta_3\eta_4(-1)^{S_1+S_2-S_3-S_4}\langle-\lambda_3-\lambda_4|T^J|-\lambda_1-\lambda_2\rangle$
C-Conservation:
$\langle\lambda_3\lambda_4|T^J|\lambda_1\lambda_2\rangle = C_3C_4(-1)^J\langle\lambda_3\lambda_4|T^J|\lambda_2\lambda_1\rangle$
(1,2=Particle/Antiparticle state;3,4=C eigen states)
T-Conservation:
$\langle\lambda_3\lambda_4|T^J|\lambda_1\lambda_2\rangle = \langle\lambda_1\lambda_2|T^J|\lambda_3\lambda_4\rangle$
Relates different processes,only relevant for elastic scattering,e.g..For first order processes($|T|^2 \ll |T|$,i.e.weak processes,eventually also e.-m.processes)
$\langle\lambda_3\lambda_4|T^J|\lambda_1\lambda_2\rangle^* = \langle\lambda_3\lambda_4|T^J|\lambda_1\lambda_2\rangle = real$

Effect on decay amplitudes:$J_M^{P(C)} \to 3+4$
P-Conservation:
$\langle\lambda_3\lambda_4|T^J|J^PM\rangle = \eta_P\eta_3\eta_4(-1)^{J-S_3-S_4}\langle -\lambda_3 - \lambda_4|T^J|J^PM\rangle$
C-Conservation:
$\langle\lambda_3\lambda_4|T^J|J^{PC}M\rangle = \eta_C(-1)^J\langle\lambda_4\lambda_3|T^J|J^{PC}M\rangle$(3,4 = Particle/Antiparticle state)
$\langle\lambda_3\lambda_4|T^J|J^{PC}M\rangle = \eta_3\eta_4\langle\lambda_3\lambda_4|T^J|J^{PC}M\rangle$(3,4 = C eigen states)
T-Conservation:
For first order processes T-conservation leads to
$\langle\lambda_3\lambda_4|T^J|J^{PC}M\rangle = \text{real}$
In addition the final state theorem([8]) can be applied:The decay amplitude of J$\to 3+4$ is related to elastic scattering amplitudes of 3+4$\to$3+4.

These rules are explicitly worked out in the following for the reaction
$\overline{p}p \to \omega\pi^0, \omega \to \gamma\pi^0$.The T-conservation plays no role here.
ω production amplitude:
$\lambda_{\overline{p}}, \lambda_p = \pm 1/2, \pm 1/2$:
$\langle 0\lambda_\omega|T^J|\pm 1/2 \pm 1/2\rangle = -\langle 0 - \lambda_\omega|T^J|\mp 1/2 \mp 1/2\rangle$ (for all J)(P-Conservation)
$\langle 0\lambda_\omega|T^J|\pm 1/2 \pm 1/2\rangle = -(-1)^J\langle 0\lambda_\omega|T^J|\pm 1/2 \pm 1/2\rangle$(C-Conservation)
$\Longrightarrow \langle 0\lambda_\omega|T^J|\pm 1/2 \pm 1/2\rangle = 0$ for J even
$\Longrightarrow \langle 0\lambda_\omega|T^J|\pm 1/2 \pm 1/2\rangle \neq 0$ for J odd
$\lambda_{\overline{p}}, \lambda_p = \pm 1/2, \mp 1/2$:
$\langle 0\lambda_\omega|T^J|\pm 1/2 \mp 1/2\rangle = -\langle 0 - \lambda_\omega|T^J|\mp 1/2 \pm 1/2\rangle$ (for all J)(P-Conservation)
$\langle 0\lambda_\omega|T^J|\pm 1/2 \mp 1/2\rangle = -(-1)^J\langle 0\lambda_\omega|T^J|\mp 1/2 \pm 1/2\rangle$(C-Conservation)
$\Longrightarrow \langle 00|T^J|\pm 1/2 \mp 1/2\rangle = 0$ for J odd(C+P-Conservation)
$\Longrightarrow \langle 00|T^J|\pm 1/2 \mp 1/2\rangle \neq 0$ for J even(C+P-Conservation)
$\Longrightarrow \langle 01|T^J|\pm 1/2 \mp 1/2\rangle = -\langle 01|T^J|\mp 1/2 \pm 1/2\rangle$ for J even(C-Conservation)
$\Longrightarrow \langle 01|T^J|\pm 1/2 \mp 1/2\rangle = \langle 01|T^J|\mp 1/2 \pm 1/2\rangle$ for J odd(C-Conservation)

ω decay amplitude:
$\langle\ 0\lambda_\gamma|T^1|1^-\rangle = -\langle\ 0 - \lambda_\gamma|T^1|1^-\rangle$(P-Conservation)

3 Spin Wave Functions and Gamma Polarization(non-rel. treatment)

Pure Spin States
Spin = 1/2
The spin wave function is given for a spin 1/2 state,e.g., by
$\psi = a|1/2\rangle + b|-1/2\rangle$.
a,b are complex numbers with $|a|^2 + |b|^2 = 1$.They determine the direction of the spin-vector.The degree of total polarization is one.Spin 1/2 is a special case.Here,only vector polarization is present and determines completely the spin state.
The spin operators are $\underline{S_x} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; $\underline{S_y} = \frac{\imath}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$; $\underline{S_z} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
$\langle S_x\rangle = \psi^\dagger \underline{S_x}\psi = 1/2(a^*b + ab^*)$
$\langle S_y\rangle = \psi^\dagger \underline{S_y}\psi = \imath/2(ab^* - a^*b)$
$\langle S_z\rangle = \psi^\dagger \underline{S_z}\psi = 1/2(a^*a - b^*b)$
The ρ matrix normalized to trρ=1 is given by
$\rho = \frac{1}{\sqrt{|a|^2+|b|^2}}\begin{pmatrix} a^*a & ab^* \\ ba^* & bb^* \end{pmatrix}$ and can be written as
$\rho = 1/2(1 + \vec{P}\vec{\sigma})$ with $\vec{\sigma}$ being the Pauli matrices.
$\vec{P} = 2\langle\vec{S}\rangle$ is the polarization vector with
$|\vec{P}|$= P =2 $Re(a^*b)^2 + 2Im(a^*b)^2 + (|a|^2 - |b|^2)$.
For pure spin 1/2 states $|\vec{P}|$ is always one.
The wave function or the ρ matrix determine completely the direction of the polarization.

Example 1:$a = 1/\sqrt{2}; b = 1/\sqrt{2}$:
$\langle S_x\rangle = 1/2; \langle S_y\rangle = 0; \langle S_z\rangle = 0$.The state is completely polarized in x-direction(P_x=1).
The ρ matrix is given by $\rho = 1/2\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$;$\vec{P} = (1,0,0); |\vec{P}| = 1$
Example 2:$a = 1/\sqrt{2}; b = \imath/\sqrt{2}$:
$\langle S_x\rangle = 0; \langle S_y\rangle = 1/2; \langle S_z\rangle = 0$.The state is completely polarized in y-direction($P_y = 1$).
The ρ matrix is given by $\rho = 1/2\begin{pmatrix} 1 & -\imath \\ \imath & -1 \end{pmatrix}$;$\vec{P} = (0,1,0); |\vec{P}| = 1$
Example 3:$a = 1/\sqrt{2}; b = 1/2(1+\imath)$:
$\langle S_x\rangle = \frac{1}{2\sqrt{2}}; \langle S_y\rangle = \frac{1}{2\sqrt{2}}; \langle S_z\rangle = 0$.The state is completely polarized in the diagonal of the x/y-direction: $\vec{P} = (1/\sqrt{2}, 1/\sqrt{2}, 0); |\vec{P}| = 1$

The ρ matrix is given by $\rho = 1/2 \begin{pmatrix} 1 & 1/\sqrt{2}(1-\imath) \\ 1/\sqrt{2}(1+\imath) & 1 \end{pmatrix}$;

Spin >1/2
For spins >1/2 the wave function has more than two components and beside the vector polarization tensor polarizations(alignment) show up.

Spin 1 is discussed more in detail in the following:
Massive Particles The spin wave function is:
$\psi = a|1\rangle + b|0\rangle + c|-1\rangle$,
$|1\rangle, |0\rangle, |-1\rangle$ are the basic spin states.
a,b,c are complex numbers and 8 real quantities (9-1 because of normalization) are necessary to fully describe the state,including its phase relations.
The ρ matrix $\rho = \frac{1}{\sqrt{|a|^2+|b|^2+|c|^2}} \begin{pmatrix} aa^* & ab^* & ac^* \\ ba^* & bb^* & bc^* \\ ca^* & cb^* & cc^* \end{pmatrix}$ contains all information available for a pure spin 1 state(tr(ρ)=1).
For spin 1 the spin operators are

$$\underline{S_x} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \underline{S_y} = \frac{\imath}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad \underline{S_z} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

For spin 1 the ρ operator can be divided in a vector-and a tensor- polarization part in a cartesian coordinate system(for details see [5])

$\rho = 1/3(\underline{1} + 3/2\vec{P}\underline{\vec{S}} + \sqrt{3/2}\sum_{i,j} T_{ij}(\underline{S_i}\underline{S_j} + \underline{S_j}\underline{S_i}))$
$=1/3(\underline{1} + 3/2\vec{P}\underline{\vec{S}} + \sqrt{3/2}\sum_{i,j} T_{ij}(\underline{S_i}\underline{S_j} + \underline{S_j}\underline{S_i} - 4/3\delta_{ij})$
$= 1/3(\underline{1} + 3/2\vec{P}\underline{\vec{S}} + 2\sum_{i,j} T_{ij}\underline{T_{ij}})$
Here $\sum_i T_{ii}$=0 has been used.
The T-matrix operators
$\underline{T_{ij}} = 1/2\sqrt{3/2}(\underline{S_i}\underline{S_j} + \underline{S_j}\underline{S_i} - 4/3 \times \underline{1})$ are given explicitly in the following

$$\underline{T_{xx}} = \begin{pmatrix} -\frac{1}{2\sqrt{6}} & 0 & \frac{\sqrt{3/2}}{2} \\ 0 & \frac{1}{\sqrt{6}} & 0 \\ \frac{\sqrt{3/2}}{2} & 0 & -\frac{1}{2\sqrt{6}} \end{pmatrix} \quad \underline{T_{yy}} = \begin{pmatrix} -\frac{1}{2\sqrt{6}} & 0 & -\frac{\sqrt{3/2}}{2} \\ 0 & \frac{1}{\sqrt{6}} & 0 \\ -\frac{\sqrt{3/2}}{2} & 0 & -\frac{1}{2\sqrt{6}} \end{pmatrix}$$

$$\underline{T_{zz}} = \begin{pmatrix} \frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & -\sqrt{2/3} & 0 \\ 0 & 0 & \frac{1}{\sqrt{6}} \end{pmatrix} \quad \underline{T_{xy}} = \begin{pmatrix} 0 & 0 & -\imath/2\sqrt{3/2} \\ 0 & 0 & 0 \\ \imath/2\sqrt{3/2} & 0 & 0 \end{pmatrix}$$

$$\underline{T_{xz}} = \begin{pmatrix} 0 & \sqrt{3}/4 & 0 \\ \sqrt{3}/4 & 0 & -\sqrt{3}/4 \\ 0 & -\sqrt{3}/4 & 0 \end{pmatrix} \quad \underline{T_{yz}} = \begin{pmatrix} 0 & -\imath\sqrt{3}/4 & 0 \\ \imath\sqrt{3}/4 & 0 & \imath\sqrt{3}/4 \\ 0 & -\imath\sqrt{3}/4 & 0 \end{pmatrix}$$

Only five of the operators are independent: $\underline{T_{xx}} + \underline{T_{yy}} + \underline{T_{zz}} = 0$

The components of $\vec{P}$ measure the vector polarization, the T_{ij} elements measure the tensor polarization.
They are given by
$\vec{P} = \mathrm{tr}(\rho\underline{\vec{S}^{\dagger}})$ and $T_{ij} = tr(\rho\underline{T_{ij}^{\dagger}}) = 1/2\sqrt{3/2}(\langle\underline{S_i}\underline{S_j} + \underline{S_j}\underline{S_i}\rangle - 4/3\delta_{ij})$

Alternatively ρ can be written in a sperical basis as

$\rho = 1/3\sum_{L,M}(2L+1)t_M^{L*}\underline{T_M^L}$ with $0 \leq L \leq 2S$ and $-L \leq M \leq L$

The multipole operators $\underline{T_M^L}$ are related to the spin matrices S_i and to the cartesian T-matrix operators $\underline{T_{ij}}$ (see [5]) and are also given below explicitly for the spin=1 case(massive particles):

$\underline{T_0^0} = \underline{1}$

$$\underline{T_0^1} = 1/\sqrt{2}\underline{S_z} = 1/\sqrt{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\underline{T_1^1} = -1/2(\underline{S_x} + \imath\underline{P_y}) = 1/\sqrt{2}\begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\underline{T_{-1}^1} = +1/2(\underline{S_x} - \imath\underline{P_y}) = 1/\sqrt{2}\begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$$

$$\underline{T_0^2} = \sqrt{3/5}\underline{T_{zz}} = \begin{pmatrix} 1/\sqrt{10} & 0 & 0 \\ 0 & -\sqrt{2/5} & 0 \\ 0 & 0 & 1/\sqrt{10} \end{pmatrix}$$

$$\underline{T_1^2} = -\sqrt{2/5}(\underline{T_{xz}} + \imath\underline{T_{yz}}) = \begin{pmatrix} 0 & -\sqrt{3/10} & 0 \\ 0 & 0 & \sqrt{3/10} \\ 0 & 0 & 0 \end{pmatrix}$$

$$\underline{T_{-1}^2} = \sqrt{2/5}(\underline{T_{xz}} - \imath\underline{T_{yz}}) = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{3/10} & 0 & 0 \\ 0 & -\sqrt{3/10} & 0 \end{pmatrix}$$

$$\underline{T_2^2} = \sqrt{1/10}(\underline{T_{xx}} - \underline{T_{yy}} + 2\imath\underline{T_{xy}}) = \begin{pmatrix} 0 & 0 & \sqrt{3/5} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\underline{T_{-2}^2} = \sqrt{1/10}(\underline{T_{xx}} - \underline{T_{yy}} - 2\imath\underline{T_{xy}}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sqrt{3/5} & 0 & 0 \end{pmatrix}$$

Note:The matrix elements of the T-operators are identical to Clebsch-Gordan coefficients: $(T_M^L)_{m,m'} = \langle sm'; LM|sm\rangle$.

The matrix operators $\underline{T_M^L}$ are measured by the complex quantities t_M^L ,which are calculated as $t_M^L = tr(\rho \underline{T_M^{L\dagger}})$

The knowledge of the t-values is equivalent to the knowledge of the ρ matrix elements(see[5],pages 54,55),e.g.
$t_0^1 = 1/\sqrt{2}(\rho_{11} - \rho_{-1-1})^*$;
$t_2^2 = \sqrt{3/5}\rho_{1-1}^*$ and so on.

For the spin 1 case,the degree of vector polarization($|\vec{P}|$) and the degree of tensor polarization(T $= \sqrt{\sum_{ij} T_{ij}^2}$) can be smaller than one,but the overall degree of polarization(d $= \sqrt{\frac{3}{4}P^2 + T^2}$) is always equal to one.

Real Photons,Mass = 0
A special spin 1 case are real photons,as here there is no helicity 0 state:
$\psi = a|1\rangle + c|-1\rangle$ and the ρ matrix has always the form $\rho^\gamma = \begin{pmatrix} x & 0 & y \\ 0 & 0 & 0 \\ x' & 0 & y' \end{pmatrix}$
T_{zz} is always equal to $\sqrt{1/6}$, corresponding to $t_0^2 = \sqrt{1/10}$.That assures that tr(ρ^γ) = 1(A pure photon state is always polarized).

A photon wave function for a pure state(elliptically polarized) can be always brought to the form

$$\psi = \frac{1}{\sqrt{2(a^2+b^2)}}(-(a+b)\exp[-\imath\Phi]|1\rangle + (a-b)\exp[\imath\Phi]|-1\rangle) \qquad (2)$$
with $a^2 + b^2 = 1$
Don't mix up the real numbers a,b defined here with the complex numbers a,b used before.Here a denotes the length of the principal axis of the polarization and Φ is angle of the principal axis,related to the x-axis.

The corresponding ρ matrix is

$$\rho = 1/2 \begin{pmatrix} 1+2a\sqrt{1-a^2}) & 0 & \exp[-2\imath\Phi](1-2a^2) \\ 0 & 0 & 0 \\ \exp[2\imath\Phi](1-2a^2) & 0 & 1-2a\sqrt{1-a^2}) \end{pmatrix}$$

As a state is always defined up to a free phase, the state (2) can also be written as
$\psi = \frac{1}{\sqrt{2(a^2+b^2)}}(-(a+b)|1\rangle + (a-b)\exp[2\imath\Phi]|-1\rangle)$
For this form of the wave function Φ is replaced in the ρ matrix by $-\Phi$.

The mode of the polarization can be determined,when the photon wave function or the photon density matrix is known.The case a $=\pm\frac{1}{\sqrt{2}}; \Phi = 0$ corresponds to pure circular polarization,the case a =1; Φ to pure linear polarization in Φ direction(see [9]).

Mixed Spin States
Often one does not deal with pure states,but with mixtures of two or more pure states.An example is the $\overline{p}p$ initial state for unpolarized particles consisting of four pure states,which appear with a probability of $p_i = 1/4$ each.The total state cannot be written as
$\psi \neq 1/4\times|1/2,1/2\rangle+1/4\times|1/2,-1/2\rangle+1/4\times|1/2,-1/2\rangle+1/4\times|-1/2,1/2\rangle$,
as p_i's are probabilities,that are connected to amplitude squares.The state is no pure state,but a mixed state,which contains incomplete information,e.g. the interference terms are missing.

The ρ matrix,however,can be formulated for mixed states
$\rho = \sum_i \rho_i = \sum_i f p_i f^\dagger$
with p_i as real numbers and $\sum_i p_i = 1$.
The ρ matrix contains the maximal available information on the state and has the same symmetry properties as the ρ matrix of a pure state.Mixed states have in general P <1(for spin 1/2 states) and d <1(for states with higher spins)
The character of a state does not change during a reaction.Thus,in unpolarized $\overline{p}p$ reactions all intermediate states(e.g. ω in the case above) and the final γ state are mixed states.

The spin density matrix of a mixed γ state can always be written as(see [5])

$$\rho = 1/2 \begin{pmatrix} 1+P_{circ} & 0 & -P_{lin}\exp[-2\imath\Phi] \\ 0 & 0 & 0 \\ -P_{lin}\exp[2\imath\Phi] & 0 & 1-P_{circ} \end{pmatrix} \quad (3)$$

with $P_{circ} = p_+ - p_-$, $P_{lin} = p_{x'} - p_{y'}$.
p_+/p_- are the probabilities to find photons with helicities +1 or -1,
$p_{x'}, p_{y'}$ are the probabilities to find the photon linearly polarized along the axis x' and y',considerably.The axis y' is perpendicular to x'.Φ is the angle of x' in relation to the x-axis(For the derivation of the formula see Appendix A).

In the following three examples for a photon wave function/density matrix for pure and mixed states are presented.
The states can be given either in a spherical or in a cartesian basis:
$|\psi\rangle = \epsilon_+|\hat{e}_+\rangle + \epsilon_-|\hat{e}_-\rangle = \epsilon_+|+1\rangle + \epsilon_-|-1\rangle$ or
$|\psi\rangle = \epsilon_x|\hat{e}_x\rangle + \epsilon_y|\hat{e}_y\rangle$ with
$\epsilon_+ = 1/\sqrt{2}(-\epsilon_x - \imath\epsilon_y); \epsilon_- = 1/\sqrt{2}(\epsilon_x - \imath\epsilon_y)$
$\epsilon_x = 1/\sqrt{2}(-\epsilon_+ + \epsilon_-); \epsilon_y = \imath/\sqrt{2}(\epsilon_+ + \epsilon_-)$
Note: For the third component(massive particles) $\hat{e}_z = \hat{e}_0$

In contrast to a pure state, P_{circ} and P_{lin} are uncorrelated and can have all values between 0 and 1.

Example 1:Circular polarized state(pure state)
$|\gamma\rangle = \hat{e}_+$
$\epsilon_+ = 1; \epsilon_- = 0$

$$\rho = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Comparison with (2) yields a=$1/\sqrt{2}$; $\Phi = 0$, i.e. a circular polarized state with helicity 1.
For the determination of the vector- and the tensor polarization components the expression $\langle A\rangle = tr(\underline{A}\rho)$ is used with $\underline{A}$ being the matrix operators of $\underline{S_x}, ..; \underline{T^2_2}, ...$:
P$_x$ =$\langle \underline{S_x}\rangle = 0$;P$_y = 0$; $P_z = 1$;t$^2_2 = t^2_{-2} = t^2_1 = t^2_{-1} = 0; t^2_0 = \sqrt{1/10}$;P $= 1$

Example 2:Linear polarized state in y-direction(pure state)
$|\gamma\rangle = \hat{e}_y$
$\epsilon_x = 0; \epsilon_y = 1; \epsilon_+ = -\imath/\sqrt{2}; \epsilon_- = -\imath/\sqrt{2}$

$$\rho = 1/2 \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Comparison with (2) yields a= 1,$\Phi = \pi/2$,i.e. a linear polarized state in y-direction.
P$_x = 0; P_y = 0; P_z = 0$;t$_2^2 = t_{-2}^2 = 1/2\sqrt{3/5}; t_1^2 = t_{-1}^2 = 0; t_0^2 = \sqrt{1/10}$;P $= 1$
Note:Linear polarization of photons has nothing to do with vector polarization,but is an effect of the tensor polarization!

Example 3:Mixture of a circular state($p_1 = 0.5$) and a linearly polarized state in y-direction($p_2 = 0.5$)(mixed state)
$|\gamma 1\rangle = \hat{e}_+; |\gamma 2\rangle = \hat{e}_y$

$$\rho = 1/2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + 1/2 \times 1/2 \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} = 1/2 \begin{pmatrix} 3/2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1/2 \end{pmatrix}$$

Comparison with (3) yields $P_{circ} = 1/2, P_{lin} =$ -1/2,i.e. an elliptically polarized photon
P$_x = 0; P_y = 0; P_z = 1/2$;t$_2^2 = t_{-2}^2 = 1/4\sqrt{3/5}; t_1^2 = t_{-1}^2 = 0; t_0^2 = \sqrt{1/10}$

4 Application of formulae to $\overline{p}p \to \omega\pi^0, \omega \to \gamma\pi^0 (\overline{p}, p$ not polarized)

The amplitude from a given initial to a given final state is
$f_{\lambda_\gamma,0,\lambda_{\overline{p}},\lambda_p}(\Theta,\theta,\phi) = \frac{1}{2p}\sum_J \sum_{\lambda_\omega}(2J+1)\sqrt{\frac{2\times1+1}{4\pi}}D^{1*}_{\lambda_\omega\lambda_\gamma}(\phi,\theta)\times$

$\times\langle 0,\lambda_\gamma \left|T^1\right| 1^-\rangle\langle 0,\lambda_\omega \left|T^J\right| \lambda_{\overline{p}},\lambda_p\rangle d^J_{\nu\mu}(\Theta)$
The ρ matrix of the final state(=ρ matrix of the gamma) is given by

$\rho^f_{\lambda_\gamma,\lambda_{\gamma'}} = \rho^\gamma_{\lambda_\gamma,\lambda_{\gamma'}} = \sum_{\lambda_{\overline{p}},\lambda_p} f_{\lambda_\gamma,0,\lambda_{\overline{p}},\lambda_p}(\Theta,\theta,\phi) f^*_{\lambda_\gamma,0,\lambda_{\overline{p}},\lambda_p}(\Theta,\theta,\phi)$

Strictly speaking,ρ^γ is the differential expression $d\rho^\gamma/dcos\Theta$.
The elements of ρ^γ are functions of the angles Θ,θ and ϕ and of the complex amplitudes $\langle\lambda_\omega, 0\left|T^J\right|\lambda_{\overline{p}},\lambda_p\rangle$ and $\langle\lambda_\gamma, 0\left|T^1\right|1^-\rangle$.
After taking into account P-and C-parity conservation,one $\langle\lambda_\gamma, 0\left|T^1\right|1^-\rangle$ =t1[J=1,λ_γ=1] amplitude is left and four(two) $\langle\lambda_\omega, 0\left|T^J\right|\lambda_{\overline{p}},\lambda_p\rangle$ = t[J,$\lambda_\omega,\lambda_{\overline{p}},\lambda_p$] amplitudes for J=odd(even). (see chapter 2)
The rectangular coordinate axes are defined as follows:
z= direction of γ momentum
x= in $\omega\pi^0$ scattering plane
y= perpendicular to $\omega\pi^0$ scattering plane

Up to J$_{max}$ = 2,ρ^γ has been calculated by Mathematica([10]).The results are discussed below.

When ρ^γ is known,all interesting physical quantities($d\sigma/d\Omega, P_{circ}, P_{lin}$ and Φ) can be calculated:
$\frac{d\sigma}{d\Omega} = tr(\rho^\gamma) = \rho^\gamma_{11} + \rho^\gamma_{-1-1}$
P_{circ}, P_{lin} and Φ can be deduced from (3):
$\frac{d\sigma}{d\Omega}\times P_{circ} = \rho^\gamma_{11} - \rho^\gamma_{-1-1}$
$\frac{d\sigma}{d\Omega}\times P_{lin} = -2\frac{Re\rho^\gamma_{1-1}}{cos2\Phi} = 2\frac{Im\rho^\gamma_{1-1}}{sin2\Phi}$
Φ = -1/2 arctan(Im ρ^γ_{1-1}/Re ρ^γ_{1-1})

These quantities can also be calculated from
$\frac{d\sigma}{d\Omega}\times P_{circ} = tr(\underline{S^\dagger_z}\rho^\gamma)$
$\frac{d\sigma}{d\Omega}\times P_{lin}\exp[-2\imath\Phi] = -tr(\underline{T^{2\dagger}_2}\rho^\gamma)$ (see [5],page 54,55)
It was checked that both methods yield the same result.
The amplitudes are given as magnitudes and phases or as real and imaginary parts.The nomenclature is

Abs/Arg[t],Abs/Arg[t1] or Re/Im[t],Re/Im[t1].
In the following some results are given explicitly.Note the two different angles Θ and θ.

Case 1:Arbitrary angles Θ, θ, ϕ /J=1,2

$$4 \times d\sigma/d\Omega = 4 \times d\sigma/dcos\Theta dcos\theta d\phi$$

```
-1/32 Abs[t1[1, 1]]^2
 (-9 Abs[t[1, 1, 1/2, -1/2]]^2 (36 + 12 Cos[2 θ] + 2 Cos[2 θ - 2 Θ] + 12 Cos[2 Θ] + 2 Cos[2 (θ + Θ)] +
      2 Cos[2 θ - 2 φ] - Cos[2 θ - 2 Θ - 2 φ] + 2 Cos[2 Θ - 2 φ] - Cos[2 (θ + Θ - φ)] - 4 Cos[2 φ] +
      2 Cos[2 (θ + φ)] - Cos[2 (θ - Θ + φ)] + 2 Cos[2 (Θ + φ)] - Cos[2 (θ + Θ + φ)]) -
   25 Abs[t[2, 1, 1/2, -1/2]]^2 (24 + 8 Cos[2 θ] + 2 Cos[2 θ - 4 Θ] + 2 Cos[2 θ - 2 Θ] + 12 Cos[2 Θ] +
      12 Cos[4 Θ] + 2 Cos[2 (θ + Θ)] + 2 Cos[2 θ + 4 Θ] + Cos[2 θ - 4 Θ - 2 φ] - Cos[2 θ - 2 Θ - 2 φ] +
      2 Cos[2 Θ - 2 φ] - 2 Cos[4 Θ - 2 φ] + Cos[2 θ + 4 Θ - 2 φ] - Cos[2 (θ + Θ - φ)] + Cos[2 (θ - 2 Θ + φ)] -
      Cos[2 (θ - Θ + φ)] + 2 Cos[2 (Θ + φ)] - Cos[2 (θ + Θ + φ)] + Cos[2 (θ + 2 Θ + φ)] - 2 Cos[4 Θ + 2 φ]) +
   30 Abs[t[1, 1, 1/2, -1/2]] (Abs[t[2, 1, 1/2, -1/2]] (12 + 4 Cos[2 θ] + 6 Cos[2 θ - 2 Θ] + 36 Cos[2 Θ] +
        6 Cos[2 (θ + Θ)] - 2 Cos[2 θ - 2 φ] + Cos[2 θ - 2 Θ - 2 φ] - 2 Cos[2 Θ - 2 φ] + Cos[2 (θ + Θ - φ)] +
        4 Cos[2 φ] - 2 Cos[2 (θ + φ)] + Cos[2 (θ - Θ + φ)] - 2 Cos[2 (Θ + φ)] + Cos[2 (θ + Θ + φ)])
      Cos[Arg[t[1, 1, 1/2, -1/2]] - Arg[t[2, 1, 1/2, -1/2]]] + 8 √3 Abs[t[2, 0, 1/2, -1/2]]
      Cos[φ] Cos[Arg[t[1, 1, 1/2, -1/2]] - Arg[t[2, 0, 1/2, -1/2]]] Sin[2 θ] Sin[2 Θ]) -
   24 (24 Abs[t[1, 0, 1/2, 1/2]]^2 Cos[Θ]^2 Sin[θ]^2 + 9 Abs[t[1, 1, 1/2, 1/2]]^2 Sin[Θ]^2 +
      3 Abs[t[1, 1, 1/2, 1/2]]^2 Cos[2 θ] Sin[Θ]^2 + 3 Abs[t[1, 1, -1/2, -1/2]]^2 (3 + Cos[2 θ]) Sin[Θ]^2 +
      12 Abs[t[1, 1, -1/2, -1/2]] Abs[t[1, 1, 1/2, 1/2]] Cos[2 φ]
       Cos[Arg[t[1, 1, -1/2, -1/2]] - Arg[t[1, 1, 1/2, 1/2]]] Sin[θ]^2 Sin[Θ]^2 + 6 Abs[t[1, 0, 1/2, 1/2]]
       Cos[φ] (Abs[t[1, 1, -1/2, -1/2]] Cos[Arg[t[1, 0, 1/2, 1/2]] - Arg[t[1, 1, -1/2, -1/2]]] -
         Abs[t[1, 1, 1/2, 1/2]] Cos[Arg[t[1, 0, 1/2, 1/2]] - Arg[t[1, 1, 1/2, 1/2]]]) Sin[2 θ] Sin[2 Θ] +
      25 Abs[t[2, 0, 1/2, -1/2]]^2 Sin[θ]^2 Sin[2 Θ]^2) - 200 √3 Abs[t[2, 0, 1/2, -1/2]]
    Abs[t[2, 1, 1/2, -1/2]] Cos[φ] Cos[Arg[t[2, 0, 1/2, -1/2]] - Arg[t[2, 1, 1/2, -1/2]]]
    Sin[2 θ] Sin[4 Θ])
```

$$4 \times d\sigma/d\Omega \times P_{circ} =$$

$$-8\,\mathrm{Abs}[\mathrm{t1}[1,1]]^2\,\mathrm{Cos}\left[\frac{\theta}{2}\right]\mathrm{Cos}\left[\frac{\theta}{2}\right]\mathrm{Cos}[\theta]\,\mathrm{Sin}\left[\frac{\theta}{2}\right]\left(\mathrm{Cos}\left[\frac{\theta}{2}\right]^2+\mathrm{Sin}\left[\frac{\theta}{2}\right]^2\right)\mathrm{Sin}\left[\frac{\theta}{2}\right]\mathrm{Sin}[\varphi]$$

$$\Bigg(9\,\mathrm{Abs}\left[t\left[1,0,\frac{1}{2},\frac{1}{2}\right]\right]\mathrm{Abs}\left[t\left[1,1,-\frac{1}{2},-\frac{1}{2}\right]\right]\mathrm{Cos}\left[\mathrm{Arg}\left[t\left[1,1,-\frac{1}{2},-\frac{1}{2}\right]\right]\right]\mathrm{Sin}\left[\mathrm{Arg}\left[t\left[1,0,\frac{1}{2},\frac{1}{2}\right]\right]\right]-$$

$$9\,\mathrm{Abs}\left[t\left[1,0,\frac{1}{2},\frac{1}{2}\right]\right]\mathrm{Abs}\left[t\left[1,1,\frac{1}{2},\frac{1}{2}\right]\right]\mathrm{Cos}\left[\mathrm{Arg}\left[t\left[1,1,\frac{1}{2},\frac{1}{2}\right]\right]\right]\mathrm{Sin}\left[\mathrm{Arg}\left[t\left[1,0,\frac{1}{2},\frac{1}{2}\right]\right]\right]-$$

$$9\,\mathrm{Abs}\left[t\left[1,0,\frac{1}{2},\frac{1}{2}\right]\right]\mathrm{Abs}\left[t\left[1,1,-\frac{1}{2},-\frac{1}{2}\right]\right]\mathrm{Cos}\left[\mathrm{Arg}\left[t\left[1,0,\frac{1}{2},\frac{1}{2}\right]\right]\right]\mathrm{Sin}\left[\mathrm{Arg}\left[t\left[1,1,-\frac{1}{2},-\frac{1}{2}\right]\right]\right]+$$

$$15\sqrt{3}\,\mathrm{Abs}\left[t\left[1,1,\frac{1}{2},-\frac{1}{2}\right]\right]\mathrm{Abs}\left[t\left[2,0,\frac{1}{2},-\frac{1}{2}\right]\right]\mathrm{Cos}\left[\frac{\theta}{2}\right]^2\mathrm{Cos}\left[\mathrm{Arg}\left[t\left[2,0,\frac{1}{2},-\frac{1}{2}\right]\right]\right]$$

$$\mathrm{Sin}\left[\mathrm{Arg}\left[t\left[1,1,\frac{1}{2},-\frac{1}{2}\right]\right]\right]+15\sqrt{3}\,\mathrm{Abs}\left[t\left[1,1,\frac{1}{2},-\frac{1}{2}\right]\right]\mathrm{Abs}\left[t\left[2,0,\frac{1}{2},-\frac{1}{2}\right]\right]$$

$$\mathrm{Cos}\left[\mathrm{Arg}\left[t\left[2,0,\frac{1}{2},-\frac{1}{2}\right]\right]\right]\mathrm{Sin}\left[\frac{\theta}{2}\right]^2\mathrm{Sin}\left[\mathrm{Arg}\left[t\left[1,1,\frac{1}{2},-\frac{1}{2}\right]\right]\right]+$$

$$9\,\mathrm{Abs}\left[t\left[1,0,\frac{1}{2},\frac{1}{2}\right]\right]\mathrm{Abs}\left[t\left[1,1,\frac{1}{2},\frac{1}{2}\right]\right]\mathrm{Cos}\left[\mathrm{Arg}\left[t\left[1,0,\frac{1}{2},\frac{1}{2}\right]\right]\right]\mathrm{Sin}\left[\mathrm{Arg}\left[t\left[1,1,\frac{1}{2},\frac{1}{2}\right]\right]\right]-$$

$$15\sqrt{3}\,\mathrm{Abs}\left[t\left[1,1,\frac{1}{2},-\frac{1}{2}\right]\right]\mathrm{Abs}\left[t\left[2,0,\frac{1}{2},-\frac{1}{2}\right]\right]\mathrm{Cos}\left[\frac{\theta}{2}\right]^2\mathrm{Cos}\left[\mathrm{Arg}\left[t\left[1,1,\frac{1}{2},-\frac{1}{2}\right]\right]\right]$$

$$\mathrm{Sin}\left[\mathrm{Arg}\left[t\left[2,0,\frac{1}{2},-\frac{1}{2}\right]\right]\right]-25\sqrt{3}\,\mathrm{Abs}\left[t\left[2,0,\frac{1}{2},-\frac{1}{2}\right]\right]\mathrm{Abs}\left[t\left[2,1,\frac{1}{2},-\frac{1}{2}\right]\right]\mathrm{Cos}\left[\frac{\theta}{2}\right]^2$$

$$\mathrm{Cos}\left[\mathrm{Arg}\left[t\left[2,1,\frac{1}{2},-\frac{1}{2}\right]\right]\right]\mathrm{Sin}\left[\mathrm{Arg}\left[t\left[2,0,\frac{1}{2},-\frac{1}{2}\right]\right]\right]+50\sqrt{3}\,\mathrm{Abs}\left[t\left[2,0,\frac{1}{2},-\frac{1}{2}\right]\right]$$

$$\mathrm{Abs}\left[t\left[2,1,\frac{1}{2},-\frac{1}{2}\right]\right]\mathrm{Cos}\left[\frac{\theta}{2}\right]^2\mathrm{Cos}[\theta]\,\mathrm{Cos}\left[\mathrm{Arg}\left[t\left[2,1,\frac{1}{2},-\frac{1}{2}\right]\right]\right]\mathrm{Sin}\left[\mathrm{Arg}\left[t\left[2,0,\frac{1}{2},-\frac{1}{2}\right]\right]\right]-$$

$$15\sqrt{3}\,\mathrm{Abs}\left[t\left[1,1,\frac{1}{2},-\frac{1}{2}\right]\right]\mathrm{Abs}\left[t\left[2,0,\frac{1}{2},-\frac{1}{2}\right]\right]\mathrm{Cos}\left[\mathrm{Arg}\left[t\left[1,1,\frac{1}{2},-\frac{1}{2}\right]\right]\right]\mathrm{Sin}\left[\frac{\theta}{2}\right]^2$$

$$\mathrm{Sin}\left[\mathrm{Arg}\left[t\left[2,0,\frac{1}{2},-\frac{1}{2}\right]\right]\right]-25\sqrt{3}\,\mathrm{Abs}\left[t\left[2,0,\frac{1}{2},-\frac{1}{2}\right]\right]\mathrm{Abs}\left[t\left[2,1,\frac{1}{2},-\frac{1}{2}\right]\right]$$

$$\mathrm{Cos}\left[\mathrm{Arg}\left[t\left[2,1,\frac{1}{2},-\frac{1}{2}\right]\right]\right]\mathrm{Sin}\left[\frac{\theta}{2}\right]^2\mathrm{Sin}\left[\mathrm{Arg}\left[t\left[2,0,\frac{1}{2},-\frac{1}{2}\right]\right]\right]-50\sqrt{3}\,\mathrm{Abs}\left[t\left[2,0,\frac{1}{2},-\frac{1}{2}\right]\right]$$

$$\mathrm{Abs}\left[t\left[2,1,\frac{1}{2},-\frac{1}{2}\right]\right]\mathrm{Cos}[\theta]\,\mathrm{Cos}\left[\mathrm{Arg}\left[t\left[2,1,\frac{1}{2},-\frac{1}{2}\right]\right]\right]\mathrm{Sin}\left[\frac{\theta}{2}\right]^2\mathrm{Sin}\left[\mathrm{Arg}\left[t\left[2,0,\frac{1}{2},-\frac{1}{2}\right]\right]\right]+$$

$$25\sqrt{3}\,\mathrm{Abs}\left[t\left[2,0,\frac{1}{2},-\frac{1}{2}\right]\right]\mathrm{Abs}\left[t\left[2,1,\frac{1}{2},-\frac{1}{2}\right]\right]\mathrm{Cos}\left[\frac{\theta}{2}\right]^2\mathrm{Cos}\left[\mathrm{Arg}\left[t\left[2,0,\frac{1}{2},-\frac{1}{2}\right]\right]\right]$$

$$\mathrm{Sin}\left[\mathrm{Arg}\left[t\left[2,1,\frac{1}{2},-\frac{1}{2}\right]\right]\right]-50\sqrt{3}\,\mathrm{Abs}\left[t\left[2,0,\frac{1}{2},-\frac{1}{2}\right]\right]\mathrm{Abs}\left[t\left[2,1,\frac{1}{2},-\frac{1}{2}\right]\right]$$

$$\mathrm{Cos}\left[\frac{\theta}{2}\right]^2\mathrm{Cos}[\theta]\,\mathrm{Cos}\left[\mathrm{Arg}\left[t\left[2,0,\frac{1}{2},-\frac{1}{2}\right]\right]\right]\mathrm{Sin}\left[\mathrm{Arg}\left[t\left[2,1,\frac{1}{2},-\frac{1}{2}\right]\right]\right]+$$

$$25\sqrt{3}\,\mathrm{Abs}\left[t\left[2,0,\frac{1}{2},-\frac{1}{2}\right]\right]\mathrm{Abs}\left[t\left[2,1,\frac{1}{2},-\frac{1}{2}\right]\right]\mathrm{Cos}\left[\mathrm{Arg}\left[t\left[2,0,\frac{1}{2},-\frac{1}{2}\right]\right]\right]\mathrm{Sin}\left[\frac{\theta}{2}\right]^2$$

$$\mathrm{Sin}\left[\mathrm{Arg}\left[t\left[2,1,\frac{1}{2},-\frac{1}{2}\right]\right]\right]+50\sqrt{3}\,\mathrm{Abs}\left[t\left[2,0,\frac{1}{2},-\frac{1}{2}\right]\right]\mathrm{Abs}\left[t\left[2,1,\frac{1}{2},-\frac{1}{2}\right]\right]$$

$$\mathrm{Cos}[\theta]\,\mathrm{Cos}\left[\mathrm{Arg}\left[t\left[2,0,\frac{1}{2},-\frac{1}{2}\right]\right]\right]\mathrm{Sin}\left[\frac{\theta}{2}\right]^2\mathrm{Sin}\left[\mathrm{Arg}\left[t\left[2,1,\frac{1}{2},-\frac{1}{2}\right]\right]\right]\Bigg)$$

$$\left(\mathrm{Cos}[\mathrm{Arg}[\mathrm{t1}[1,1]]]^2+\mathrm{Sin}[\mathrm{Arg}[\mathrm{t1}[1,1]]]^2\right)$$

$$4 \times d\sigma/d\Omega \times P_{lin} cos2\Phi =$$

$$-\frac{1}{64}\,\mathrm{Abs}[\mathrm{t1}[1,1]]^2 \left(-288\,\mathrm{Abs}\left[t\left[1,1,\frac{1}{2},-\frac{1}{2}\right]\right]^2\right.$$

$$\left(\mathrm{Cos}\left[\frac{\theta}{2}\right]^4 \mathrm{Cos}[2\varphi]\,\mathrm{Sin}[\theta]^2 + \mathrm{Cos}[2\varphi]\,\mathrm{Sin}[\theta]^2\,\mathrm{Sin}\left[\frac{\theta}{2}\right]^4 - \left(\mathrm{Cos}\left[\frac{\theta}{2}\right]^4 + \mathrm{Cos}[4\varphi]\,\mathrm{Sin}\left[\frac{\theta}{2}\right]^4\right)\mathrm{Sin}[\Theta]^2\right) + 15$$

$$\mathrm{Abs}\left[t\left[1,1,\frac{1}{2},-\frac{1}{2}\right]\right]$$

$$\left(-\mathrm{Abs}\left[t\left[2,1,\frac{1}{2},-\frac{1}{2}\right]\right]\,(-12 - 16\,\mathrm{Cos}[\theta] - 4\,\mathrm{Cos}[2\theta] + 8\,\mathrm{Cos}[\theta - 2\Theta] + 2\,\mathrm{Cos}[2\theta - 2\Theta] + 12\,\mathrm{Cos}[2\Theta] +\right.$$

$$2\,\mathrm{Cos}[2\,(\theta+\Theta)] + 8\,\mathrm{Cos}[\theta+2\Theta] + 8\,\mathrm{Cos}[\theta-4\varphi] - 2\,\mathrm{Cos}[2\theta-4\varphi] - 4\,\mathrm{Cos}[\theta-2\Theta-4\varphi] +$$

$$\mathrm{Cos}[2\theta-2\Theta-4\varphi] + 6\,\mathrm{Cos}[2\Theta-4\varphi] - 4\,\mathrm{Cos}[\theta+2\Theta-4\varphi] + 4\,\mathrm{Cos}[2\theta-2\varphi] + 6\,\mathrm{Cos}[2\theta-2\Theta-2\varphi] +$$

$$\mathrm{Cos}[2\,(\theta+\Theta-2\varphi)] - 12\,\mathrm{Cos}[2\Theta-2\varphi] + 6\,\mathrm{Cos}[2\,(\theta+\Theta-\varphi)] - 8\,\mathrm{Cos}[2\varphi] - 12\,\mathrm{Cos}[4\varphi] +$$

$$4\,\mathrm{Cos}[2\,(\theta+\varphi)] + 6\,\mathrm{Cos}[2\,(\theta-\Theta+\varphi)] - 12\,\mathrm{Cos}[2\,(\Theta+\varphi)] + 6\,\mathrm{Cos}[2\,(\theta+\Theta+\varphi)] +$$

$$\mathrm{Cos}[2\,(\theta+\Theta+2\varphi)] + 8\,\mathrm{Cos}[\theta+4\varphi] - 2\,\mathrm{Cos}[2\theta+4\varphi] - 4\,\mathrm{Cos}[\theta-2\Theta+4\varphi] + \mathrm{Cos}[2\theta-2\Theta+4\varphi] +$$

$$6\,\mathrm{Cos}[2\Theta+4\varphi] - 4\,\mathrm{Cos}[\theta+2\Theta+4\varphi])\,\mathrm{Cos}\left[\mathrm{Arg}\left[t\left[1,1,\frac{1}{2},-\frac{1}{2}\right]\right] - \mathrm{Arg}\left[t\left[2,1,\frac{1}{2},-\frac{1}{2}\right]\right]\right] -$$

$$16\sqrt{3}\,\mathrm{Abs}\left[t\left[2,0,\frac{1}{2},-\frac{1}{2}\right]\right]\,\mathrm{Cos}[\varphi]\,(2 + \mathrm{Cos}[\theta-2\varphi] - 2\,\mathrm{Cos}[2\varphi] + \mathrm{Cos}[\theta+2\varphi])$$

$$\left.\mathrm{Cos}\left[\mathrm{Arg}\left[t\left[1,1,\frac{1}{2},-\frac{1}{2}\right]\right] - \mathrm{Arg}\left[t\left[2,0,\frac{1}{2},-\frac{1}{2}\right]\right]\right]\,\mathrm{Sin}[\theta]\,\mathrm{Sin}[2\Theta]\right) +$$

$$4\left(\frac{25}{8}\,\mathrm{Abs}\left[t\left[2,1,\frac{1}{2},-\frac{1}{2}\right]\right]^2 (8\,\mathrm{Cos}[\theta-4\Theta] + 2\,\mathrm{Cos}[2\theta-4\Theta] - 8\,\mathrm{Cos}[\theta-2\Theta] - 2\,\mathrm{Cos}[2\theta-2\Theta] -\right.$$

$$12\,\mathrm{Cos}[2\Theta] + 12\,\mathrm{Cos}[4\Theta] - 2\,\mathrm{Cos}[2\,(\theta+\Theta)] - 8\,\mathrm{Cos}[\theta+2\Theta] + 8\,\mathrm{Cos}[\theta+4\Theta] + 2\,\mathrm{Cos}[2\theta+4\Theta] +$$

$$4\,\mathrm{Cos}[\theta-2\Theta-4\varphi] - \mathrm{Cos}[2\theta-2\Theta-4\varphi] - 6\,\mathrm{Cos}[2\Theta-4\varphi] + 4\,\mathrm{Cos}[\theta+2\Theta-4\varphi] + 6\,\mathrm{Cos}[4\Theta-4\varphi] -$$

$$4\,\mathrm{Cos}[\theta+4\Theta-4\varphi] + \mathrm{Cos}[2\theta+4\Theta-4\varphi] + 8\,\mathrm{Cos}[2\Theta-2\varphi] + 2\,\mathrm{Cos}[2\theta-4\Theta-2\varphi] +$$

$$2\,\mathrm{Cos}[2\theta-2\Theta-2\varphi] - \mathrm{Cos}[2\,(\theta+\Theta-2\varphi)] - 4\,\mathrm{Cos}[2\Theta-2\varphi] - 4\,\mathrm{Cos}[4\Theta-2\varphi] +$$

$$2\,\mathrm{Cos}[2\theta+4\Theta-2\varphi] + 2\,\mathrm{Cos}[2\,(\theta+\Theta-\varphi)] - 16\,\mathrm{Cos}[2\varphi] + 8\,\mathrm{Cos}[2\,(\Theta+\varphi)] + 2\,\mathrm{Cos}[2\,(\theta-2\Theta+\varphi)] +$$

$$2\,\mathrm{Cos}[2\,(\theta-\Theta+\varphi)] - 4\,\mathrm{Cos}[2\,(\Theta+\varphi)] + 6\,\mathrm{Cos}[4\,(\Theta+\varphi)] + 2\,\mathrm{Cos}[2\,(\theta+\Theta+\varphi)] +$$

$$2\,\mathrm{Cos}[2\,(\theta+2\Theta+\varphi)] - \mathrm{Cos}[2\,(\theta+\Theta+2\varphi)] - 4\,\mathrm{Cos}[4\Theta+2\varphi] - 4\,\mathrm{Cos}[\theta-4\Theta+4\varphi] + \mathrm{Cos}[$$

$$2\theta-4\Theta+4\varphi] + 4\,\mathrm{Cos}[\theta-2\Theta+4\varphi] - \mathrm{Cos}[2\theta-2\Theta+4\varphi] - 6\,\mathrm{Cos}[2\Theta+4\varphi] + 4\,\mathrm{Cos}[\theta+2\Theta+4\varphi] -$$

$$4\,\mathrm{Cos}[\theta-4\,(\Theta+\varphi)] + \mathrm{Cos}[2\,(\theta-2\,(\Theta+\varphi))] + \mathrm{Cos}[2\,(\theta+2\,(\Theta+\varphi))] - 4\,\mathrm{Cos}[\theta+4\,(\Theta+\varphi)]) +$$

$$144\,\mathrm{Abs}\left[t\left[1,0,\frac{1}{2},\frac{1}{2}\right]\right]^2 \mathrm{Cos}[\Theta]^2\,\mathrm{Cos}[2\varphi]\,\mathrm{Sin}[\theta]^2 -$$

$$3\left(3\,\mathrm{Abs}\left[t\left[1,1,-\frac{1}{2},-\frac{1}{2}\right]\right]\,\mathrm{Abs}\left[t\left[1,1,\frac{1}{2},\frac{1}{2}\right]\right]\,(6 + 8\,\mathrm{Cos}[\theta] + 2\,\mathrm{Cos}[2\theta] -\right.$$

$$4\,\mathrm{Cos}[\theta-4\varphi] + \mathrm{Cos}[2\theta-4\varphi] + 6\,\mathrm{Cos}[4\varphi] - 4\,\mathrm{Cos}[\theta+4\varphi] + \mathrm{Cos}[2\theta+4\varphi])\,\mathrm{Cos}[$$

$$\mathrm{Arg}\left[t\left[1,1,-\frac{1}{2},-\frac{1}{2}\right]\right] - \mathrm{Arg}\left[t\left[1,1,\frac{1}{2},\frac{1}{2}\right]\right]] + 12\,\mathrm{Abs}\left[t\left[1,1,-\frac{1}{2},-\frac{1}{2}\right]\right]^2 \mathrm{Cos}[2\varphi]\,\mathrm{Sin}[\theta]^2 +$$

$$\left.4\left(3\,\mathrm{Abs}\left[t\left[1,1,\frac{1}{2},\frac{1}{2}\right]\right]^2 - 50\,\mathrm{Abs}\left[t\left[2,0,\frac{1}{2},-\frac{1}{2}\right]\right]^2 \mathrm{Cos}[\Theta]^2\right)\mathrm{Cos}[2\varphi]\,\mathrm{Sin}[\theta]^2\right)\mathrm{Sin}[\Theta]^2 +$$

$$36\,\mathrm{Abs}\left[t\left[1,0,\frac{1}{2},\frac{1}{2}\right]\right]\,\mathrm{Cos}[\varphi]\,(2 + \mathrm{Cos}[\theta-2\varphi] - 2\,\mathrm{Cos}[2\varphi] + \mathrm{Cos}[\theta+2\varphi])$$

$$\left(\mathrm{Abs}\left[t\left[1,1,-\frac{1}{2},-\frac{1}{2}\right]\right]\,\mathrm{Cos}\left[\mathrm{Arg}\left[t\left[1,0,\frac{1}{2},\frac{1}{2}\right]\right] - \mathrm{Arg}\left[t\left[1,1,-\frac{1}{2},-\frac{1}{2}\right]\right]\right] -\right.$$

$$\left.\mathrm{Abs}\left[t\left[1,1,\frac{1}{2},\frac{1}{2}\right]\right]\,\mathrm{Cos}\left[\mathrm{Arg}\left[t\left[1,0,\frac{1}{2},\frac{1}{2}\right]\right] - \mathrm{Arg}\left[t\left[1,1,\frac{1}{2},\frac{1}{2}\right]\right]\right]\right)\mathrm{Sin}[\theta]\,\mathrm{Sin}[2\Theta] + 50\sqrt{3}$$

$$\mathrm{Abs}\left[t\left[2,0,\frac{1}{2},-\frac{1}{2}\right]\right]\,\mathrm{Abs}\left[t\left[2,1,\frac{1}{2},-\frac{1}{2}\right]\right]\,\mathrm{Cos}[\varphi]\,(2 + \mathrm{Cos}[\theta-2\varphi] - 2\,\mathrm{Cos}[2\varphi] + \mathrm{Cos}[\theta+2\varphi])$$

$$\left.\left.\mathrm{Cos}\left[\mathrm{Arg}\left[t\left[2,0,\frac{1}{2},-\frac{1}{2}\right]\right] - \mathrm{Arg}\left[t\left[2,1,\frac{1}{2},-\frac{1}{2}\right]\right]\right]\,\mathrm{Sin}[\theta]\,\mathrm{Sin}[4\Theta]\right)\right)$$

The output for Φ is rather long and is not explicitly given here.
The same holds for the output in terms of real and imaginary parts of the amplitudes.

Case2: $\Theta = 0, \quad \theta, \phi$ arbitrary $\quad$ /J=1,2

For comparison,here the results are given in the form Abs/Arg and in the form Re/Im.

$4 \times d\sigma/d\Omega = 4 \times d\sigma/dcos\Theta dcos\theta d\phi =$

$$-\frac{1}{2}\,\mathrm{Abs}[t1[1,1]]^2 \Big(-(3+\mathrm{Cos}[2\theta]) \Big(9\,\mathrm{Abs}\big[t\big[1,1,\tfrac{1}{2},-\tfrac{1}{2}\big]\big]^2 + 25\,\mathrm{Abs}\big[t\big[2,1,\tfrac{1}{2},-\tfrac{1}{2}\big]\big]^2 - 30\,\mathrm{Abs}\big[t\big[1,1,\tfrac{1}{2},-\tfrac{1}{2}\big]\big]\,\mathrm{Abs}\big[t\big[2,1,\tfrac{1}{2},-\tfrac{1}{2}\big]\big] \mathrm{Cos}\big[\mathrm{Arg}\big[t\big[1,1,\tfrac{1}{2},-\tfrac{1}{2}\big]\big] - \mathrm{Arg}\big[t\big[2,1,\tfrac{1}{2},-\tfrac{1}{2}\big]\big]\big]\Big) - 36\,\mathrm{Abs}\big[t\big[1,0,\tfrac{1}{2},\tfrac{1}{2}\big]\big]^2\,\mathrm{Sin}[\theta]^2\Big)$$

$$= \frac{1}{2}\left(\mathrm{Im}[t1[1,1]]^2 + \mathrm{Re}[t1[1,1]]^2\right) \Big(9\,(3+\mathrm{Cos}[2\theta])\,\mathrm{Im}\big[t\big[1,1,\tfrac{1}{2},-\tfrac{1}{2}\big]\big]^2 - 30\,(3+\mathrm{Cos}[2\theta])\,\mathrm{Im}\big[t\big[1,1,\tfrac{1}{2},-\tfrac{1}{2}\big]\big]\,\mathrm{Im}\big[t\big[2,1,\tfrac{1}{2},-\tfrac{1}{2}\big]\big] + 25\,(3+\mathrm{Cos}[2\theta])\,\mathrm{Im}\big[t\big[2,1,\tfrac{1}{2},-\tfrac{1}{2}\big]\big]^2 + 18\,\mathrm{Re}\big[t\big[1,0,\tfrac{1}{2},\tfrac{1}{2}\big]\big]^2 - 18\,\mathrm{Cos}[2\theta]\,\mathrm{Re}\big[t\big[1,0,\tfrac{1}{2},\tfrac{1}{2}\big]\big]^2 + 27\,\mathrm{Re}\big[t\big[1,1,\tfrac{1}{2},-\tfrac{1}{2}\big]\big]^2 + 9\,\mathrm{Cos}[2\theta]\,\mathrm{Re}\big[t\big[1,1,\tfrac{1}{2},-\tfrac{1}{2}\big]\big]^2 - 90\,\mathrm{Re}\big[t\big[1,1,\tfrac{1}{2},-\tfrac{1}{2}\big]\big]\,\mathrm{Re}\big[t\big[2,1,\tfrac{1}{2},-\tfrac{1}{2}\big]\big] - 30\,\mathrm{Cos}[2\theta]\,\mathrm{Re}\big[t\big[1,1,\tfrac{1}{2},-\tfrac{1}{2}\big]\big]\,\mathrm{Re}\big[t\big[2,1,\tfrac{1}{2},-\tfrac{1}{2}\big]\big] + 75\,\mathrm{Re}\big[t\big[2,1,\tfrac{1}{2},-\tfrac{1}{2}\big]\big]^2 + 25\,\mathrm{Cos}[2\theta]\,\mathrm{Re}\big[t\big[2,1,\tfrac{1}{2},-\tfrac{1}{2}\big]\big]^2 + 36\,\mathrm{Im}\big[t\big[1,0,\tfrac{1}{2},\tfrac{1}{2}\big]\big]^2\,\mathrm{Sin}[\theta]^2\Big)$$

$4 \times d\sigma/d\Omega \times P_{circ} = 0$

$4 \times d\sigma/d\Omega \times P_{lin} cos2\Phi =$

$$-\frac{1}{2}\,\mathrm{Abs}[t1[1,1]]^2\,\mathrm{Cos}[2\varphi] \Big(18\,\mathrm{Abs}\big[t\big[1,0,\tfrac{1}{2},\tfrac{1}{2}\big]\big]^2 - 9\,\mathrm{Abs}\big[t\big[1,1,\tfrac{1}{2},-\tfrac{1}{2}\big]\big]^2 - 25\,\mathrm{Abs}\big[t\big[2,1,\tfrac{1}{2},-\tfrac{1}{2}\big]\big]^2 + 30\,\mathrm{Abs}\big[t\big[1,1,\tfrac{1}{2},-\tfrac{1}{2}\big]\big]\,\mathrm{Abs}\big[t\big[2,1,\tfrac{1}{2},-\tfrac{1}{2}\big]\big]\,\mathrm{Cos}\big[\mathrm{Arg}\big[t\big[1,1,\tfrac{1}{2},-\tfrac{1}{2}\big]\big] - \mathrm{Arg}\big[t\big[2,1,\tfrac{1}{2},-\tfrac{1}{2}\big]\big]\big]\Big)\,\mathrm{Sin}[\theta]^2$$

$$= -\frac{1}{2}\,\mathrm{Cos}[2\varphi] \Big(18\,\mathrm{Im}\big[t\big[1,0,\tfrac{1}{2},\tfrac{1}{2}\big]\big]^2 - 9\,\mathrm{Im}\big[t\big[1,1,\tfrac{1}{2},-\tfrac{1}{2}\big]\big]^2 + 30\,\mathrm{Im}\big[t\big[1,1,\tfrac{1}{2},-\tfrac{1}{2}\big]\big]\,\mathrm{Im}\big[t\big[2,1,\tfrac{1}{2},-\tfrac{1}{2}\big]\big] - 25\,\mathrm{Im}\big[t\big[2,1,\tfrac{1}{2},-\tfrac{1}{2}\big]\big]^2 + 18\,\mathrm{Re}\big[t\big[1,0,\tfrac{1}{2},\tfrac{1}{2}\big]\big]^2 - 9\,\mathrm{Re}\big[t\big[1,1,\tfrac{1}{2},-\tfrac{1}{2}\big]\big]^2 + 30\,\mathrm{Re}\big[t\big[1,1,\tfrac{1}{2},-\tfrac{1}{2}\big]\big]\,\mathrm{Re}\big[t\big[2,1,\tfrac{1}{2},-\tfrac{1}{2}\big]\big] - 25\,\mathrm{Re}\big[t\big[2,1,\tfrac{1}{2},-\tfrac{1}{2}\big]\big]^2\Big) \left(\mathrm{Im}[t1[1,1]]^2 + \mathrm{Re}[t1[1,1]]^2\right)\mathrm{Sin}[\theta]^2$$

$\Phi = -\phi$

Case 3:θ=0, Θ, ϕ arbitrary /J=1,2

Here the results are only given in terms of Re/Im-parts of the amplitudes.

$4 \times d\sigma/dcos\Theta dcos\theta d\phi =$

$$
\begin{aligned}
&\frac{1}{2}\left(\mathrm{Im}[t1[1,1]]^2+\mathrm{Re}[t1[1,1]]^2\right)\\
&\Big(9\,(3+\mathrm{Cos}[2\theta])\,\mathrm{Im}\left[t\left[1,1,\tfrac{1}{2},-\tfrac{1}{2}\right]\right]^2-30\,(1+3\,\mathrm{Cos}[2\theta])\,\mathrm{Im}\left[t\left[1,1,\tfrac{1}{2},-\tfrac{1}{2}\right]\right]\mathrm{Im}\left[t\left[2,1,\tfrac{1}{2},-\tfrac{1}{2}\right]\right]+\\
&\quad 25\,(2+\mathrm{Cos}[2\theta]+\mathrm{Cos}[4\theta])\,\mathrm{Im}\left[t\left[2,1,\tfrac{1}{2},-\tfrac{1}{2}\right]\right]^2+9\,\mathrm{Re}\left[t\left[1,1,-\tfrac{1}{2},-\tfrac{1}{2}\right]\right]^2-\\
&\quad 9\,\mathrm{Cos}[2\theta]\,\mathrm{Re}\left[t\left[1,1,-\tfrac{1}{2},-\tfrac{1}{2}\right]\right]^2+27\,\mathrm{Re}\left[t\left[1,1,\tfrac{1}{2},-\tfrac{1}{2}\right]\right]^2+9\,\mathrm{Cos}[2\theta]\,\mathrm{Re}\left[t\left[1,1,\tfrac{1}{2},-\tfrac{1}{2}\right]\right]^2+\\
&\quad 9\,\mathrm{Re}\left[t\left[1,1,\tfrac{1}{2},\tfrac{1}{2}\right]\right]^2-9\,\mathrm{Cos}[2\theta]\,\mathrm{Re}\left[t\left[1,1,\tfrac{1}{2},\tfrac{1}{2}\right]\right]^2-30\,\mathrm{Re}\left[t\left[1,1,\tfrac{1}{2},-\tfrac{1}{2}\right]\right]\mathrm{Re}\left[t\left[2,1,\tfrac{1}{2},-\tfrac{1}{2}\right]\right]-\\
&\quad 90\,\mathrm{Cos}[2\theta]\,\mathrm{Re}\left[t\left[1,1,\tfrac{1}{2},-\tfrac{1}{2}\right]\right]\mathrm{Re}\left[t\left[2,1,\tfrac{1}{2},-\tfrac{1}{2}\right]\right]+50\,\mathrm{Re}\left[t\left[2,1,\tfrac{1}{2},-\tfrac{1}{2}\right]\right]^2+\\
&\quad 25\,\mathrm{Cos}[2\theta]\,\mathrm{Re}\left[t\left[2,1,\tfrac{1}{2},-\tfrac{1}{2}\right]\right]^2+25\,\mathrm{Cos}[4\theta]\,\mathrm{Re}\left[t\left[2,1,\tfrac{1}{2},-\tfrac{1}{2}\right]\right]^2+\\
&\quad 18\,\mathrm{Im}\left[t\left[1,1,-\tfrac{1}{2},-\tfrac{1}{2}\right]\right]^2\mathrm{Sin}[\theta]^2+18\,\mathrm{Im}\left[t\left[1,1,\tfrac{1}{2},\tfrac{1}{2}\right]\right]^2\mathrm{Sin}[\theta]^2\Big)
\end{aligned}
$$

$4 \times d\sigma/d\Omega \times P_{circ} = 0$

$4 \times d\sigma/d\Omega \times P_{lin} cos2\Phi =$

$$
\begin{aligned}
&\frac{1}{2}\Big(-9\,\mathrm{Im}\left[t\left[1,1,\tfrac{1}{2},-\tfrac{1}{2}\right]\right]^2+18\,\mathrm{Im}\left[t\left[1,1,-\tfrac{1}{2},-\tfrac{1}{2}\right]\right]\mathrm{Im}\left[t\left[1,1,\tfrac{1}{2},\tfrac{1}{2}\right]\right]-\\
&\quad 30\,\mathrm{Im}\left[t\left[1,1,\tfrac{1}{2},-\tfrac{1}{2}\right]\right]\mathrm{Im}\left[t\left[2,1,\tfrac{1}{2},-\tfrac{1}{2}\right]\right]+25\,\mathrm{Im}\left[t\left[2,1,\tfrac{1}{2},-\tfrac{1}{2}\right]\right]^2+\\
&\quad 50\,\mathrm{Cos}[2\theta]\,\mathrm{Im}\left[t\left[2,1,\tfrac{1}{2},-\tfrac{1}{2}\right]\right]^2-9\,\mathrm{Re}\left[t\left[1,1,\tfrac{1}{2},-\tfrac{1}{2}\right]\right]^2+\\
&\quad 18\,\mathrm{Re}\left[t\left[1,1,-\tfrac{1}{2},-\tfrac{1}{2}\right]\right]\mathrm{Re}\left[t\left[1,1,\tfrac{1}{2},\tfrac{1}{2}\right]\right]-30\,\mathrm{Re}\left[t\left[1,1,\tfrac{1}{2},-\tfrac{1}{2}\right]\right]\mathrm{Re}\left[t\left[2,1,\tfrac{1}{2},-\tfrac{1}{2}\right]\right]+\\
&\quad 25\,\mathrm{Re}\left[t\left[2,1,\tfrac{1}{2},-\tfrac{1}{2}\right]\right]^2+50\,\mathrm{Cos}[2\theta]\,\mathrm{Re}\left[t\left[2,1,\tfrac{1}{2},-\tfrac{1}{2}\right]\right]^2\Big)\left(\mathrm{Im}[t1[1,1]]^2+\mathrm{Re}[t1[1,1]]^2\right)\mathrm{Sin}[\theta]^2
\end{aligned}
$$

$\Phi = 0$

General observations:
Case 1: Θ, θ, ϕ arbitrary
-All observables are proportional to $|t1|^2$
-dσ/dΩ and P_{lin} are functions of six |amplitudes| (as expected) and of six terms with cos$\Delta\phi$,where $\Delta\phi$ are differences between the phases of the amplitudes.
Written as Re/Im-parts of amplitudes only products of Re/Re- or Im/Im-parts of the amplitudes appear,no mixed terms.
-P_{circ} is a function of six |amplitudes| and of the products of phases of the contributing amplitudes.
Written as Re/Im parts of amplitudes only mixed terms of amplitudes(Re$\times$Im) appear.
That means that the measurement of P_{circ} yields additional information,not available from the measurement of dσ/dΩ and P_{lin}.
-P_{lin} expressing the tensor polarization of the gamma is always present,except for special combinations of angles and amplitudes.

Case 2:$\Theta = 0, \quad \theta, \phi$ arbitrary
Here P_{circ} is always $= 0$.
That correspond to the fact,that ρ^{ω}_{10} and ρ^{ω}_{1-1} are zero for $\Theta = 0,\pi$(see also [5],pages 99,100).

Case 3:$\theta = 0, \quad \Theta, \phi$ arbitrary
Here $P_{circ} = 0$ and $\Phi = 0$.
That can be understood in a classical picture:
ω has a vector polarization in y-direction,created by a circular movement of a charge in the x-z plane.Then in $\theta = 0$ direction,$P_{circ} = 0$ and the linear polarization is in x-direction($\Phi = 0$).

The calculations done here with ρ^{γ} were also performed in a two-step process by firstly evaluating ρ^{ω} and then calculating ρ^{γ} as $\rho^{\gamma} = f_{\omega\rightarrow\pi^0\gamma}\rho^{\omega}f^{\dagger}_{\omega\rightarrow\pi^0\gamma}$.Here it becomes obvious,how the polarization effects depend on the ρ_{ω} matrix elements.
For instance,
$P_{circ} =$ - $2\sqrt{2}$ Im$\rho_{1,0}$ sinθ sinϕ,
showing that Im$\rho_{1,0}$ is responsible for the gamma vector polarization for $\theta \neq 0$ and $\phi \neq 0$.

5 Comparison $\omega \to \gamma\pi^0/\omega \to \pi^+\pi^-\pi^0$

ρ^f was also calculated with Mathematica for the decay $\omega \to \pi^+\pi^-\pi^0$.Here only the differential cross section is relevant(no spins in the final state).Some examples are given below.

Case 1:Θ, θ, ϕ arbitrary /J =1,2

$4 \times d\sigma/d\Omega =$

$$\frac{1}{8}\mathrm{Abs}[t1[1,0]]^2 \Big(144\,\mathrm{Abs}\left[t\left[1,0,\tfrac{1}{2},\tfrac{1}{2}\right]\right]^2 \mathrm{Cos}[\theta]^2\,\mathrm{Cos}[\Theta]^2 - 9\,\mathrm{Abs}\left[t\left[1,1,\tfrac{1}{2},-\tfrac{1}{2}\right]\right]^2$$
$$(-6-2\,\mathrm{Cos}[2\theta]+\mathrm{Cos}[2\theta-2\varphi]-2\,\mathrm{Cos}[2\varphi]+\mathrm{Cos}[2(\theta+\varphi)])\,\mathrm{Sin}[\theta]^2 - 36\,\mathrm{Abs}\left[t\left[1,0,\tfrac{1}{2},\tfrac{1}{2}\right]\right]\mathrm{Cos}[\varphi]$$
$$\Big(\mathrm{Abs}\left[t\left[1,1,-\tfrac{1}{2},-\tfrac{1}{2}\right]\right]\mathrm{Cos}\left[\mathrm{Arg}\left[t\left[1,0,\tfrac{1}{2},\tfrac{1}{2}\right]\right]-\mathrm{Arg}\left[t\left[1,1,-\tfrac{1}{2},-\tfrac{1}{2}\right]\right]\right] - \mathrm{Abs}\left[t\left[1,1,\tfrac{1}{2},\tfrac{1}{2}\right]\right]$$
$$\mathrm{Cos}\left[\mathrm{Arg}\left[t\left[1,0,\tfrac{1}{2},\tfrac{1}{2}\right]\right]-\mathrm{Arg}\left[t\left[1,1,\tfrac{1}{2},\tfrac{1}{2}\right]\right]\right]\Big)\mathrm{Sin}[2\theta]\,\mathrm{Sin}[2\Theta] - 30\,\mathrm{Abs}\left[t\left[1,1,\tfrac{1}{2},-\tfrac{1}{2}\right]\right]$$
$$\Big(\mathrm{Abs}\left[t\left[2,1,\tfrac{1}{2},-\tfrac{1}{2}\right]\right](2+6\,\mathrm{Cos}[2\theta]+\mathrm{Cos}[2\theta-2\varphi]-2\,\mathrm{Cos}[2\varphi]+\mathrm{Cos}[2(\theta+\varphi)])$$
$$\mathrm{Cos}\left[\mathrm{Arg}\left[t\left[1,1,\tfrac{1}{2},-\tfrac{1}{2}\right]\right]-\mathrm{Arg}\left[t\left[2,1,\tfrac{1}{2},-\tfrac{1}{2}\right]\right]\right]\mathrm{Sin}[\theta]^2 - 2\sqrt{3}\,\mathrm{Abs}\left[t\left[2,0,\tfrac{1}{2},-\tfrac{1}{2}\right]\right]$$
$$\mathrm{Cos}[\varphi]\,\mathrm{Cos}\left[\mathrm{Arg}\left[t\left[1,1,\tfrac{1}{2},-\tfrac{1}{2}\right]\right]-\mathrm{Arg}\left[t\left[2,0,\tfrac{1}{2},-\tfrac{1}{2}\right]\right]\right]\mathrm{Sin}[2\theta]\,\mathrm{Sin}[2\Theta]\Big) +$$
$$2\Big(\frac{25}{2}\mathrm{Abs}\left[t\left[2,1,\tfrac{1}{2},-\tfrac{1}{2}\right]\right]^2(4+2\,\mathrm{Cos}[2\theta]+2\,\mathrm{Cos}[4\Theta]-\mathrm{Cos}[2\theta-2\varphi]+$$
$$\mathrm{Cos}[4\Theta-2\varphi]-\mathrm{Cos}[2(\theta+\varphi)]+\mathrm{Cos}[4\Theta+2\varphi])\,\mathrm{Sin}[\theta]^2+$$
$$3\Big(6\Big(\mathrm{Abs}\left[t\left[1,1,-\tfrac{1}{2},-\tfrac{1}{2}\right]\right]^2+\mathrm{Abs}\left[t\left[1,1,\tfrac{1}{2},\tfrac{1}{2}\right]\right]^2-2\,\mathrm{Abs}\left[t\left[1,1,-\tfrac{1}{2},-\tfrac{1}{2}\right]\right]\mathrm{Abs}\left[t\left[1,1,\tfrac{1}{2},\tfrac{1}{2}\right]\right]$$
$$\mathrm{Cos}[2\varphi]\,\mathrm{Cos}\left[\mathrm{Arg}\left[t\left[1,1,-\tfrac{1}{2},-\tfrac{1}{2}\right]\right]-\mathrm{Arg}\left[t\left[1,1,\tfrac{1}{2},\tfrac{1}{2}\right]\right]\right]\Big)\mathrm{Sin}[\theta]^2\,\mathrm{Sin}[\Theta]^2+$$
$$25\,\mathrm{Abs}\left[t\left[2,0,\tfrac{1}{2},-\tfrac{1}{2}\right]\right]^2\mathrm{Cos}[\theta]^2\,\mathrm{Sin}[2\Theta]^2\Big) - 25\sqrt{3}\,\mathrm{Abs}\left[t\left[2,0,\tfrac{1}{2},-\tfrac{1}{2}\right]\right]\mathrm{Abs}\left[t\left[2,1,\tfrac{1}{2},-\tfrac{1}{2}\right]\right]$$
$$\mathrm{Cos}[\varphi]\,\mathrm{Cos}\left[\mathrm{Arg}\left[t\left[2,0,\tfrac{1}{2},-\tfrac{1}{2}\right]\right]-\mathrm{Arg}\left[t\left[2,1,\tfrac{1}{2},-\tfrac{1}{2}\right]\right]\right]\mathrm{Sin}[2\theta]\,\mathrm{Sin}[4\Theta]\Big)\Big)$$

Case 2:$\Theta = 0$, θ, ϕ arbitrary /J =1,2

$4 \times d\sigma/d\Omega =$

$$\frac{1}{2}\mathrm{Abs}[t1[1,0]]^2\Big(36\,\mathrm{Abs}\left[t\left[1,0,\tfrac{1}{2},\tfrac{1}{2}\right]\right]^2\mathrm{Cos}[\theta]^2+$$
$$2\Big(9\,\mathrm{Abs}\left[t\left[1,1,\tfrac{1}{2},-\tfrac{1}{2}\right]\right]^2+25\,\mathrm{Abs}\left[t\left[2,1,\tfrac{1}{2},-\tfrac{1}{2}\right]\right]^2-30\,\mathrm{Abs}\left[t\left[1,1,\tfrac{1}{2},-\tfrac{1}{2}\right]\right]$$
$$\mathrm{Abs}\left[t\left[2,1,\tfrac{1}{2},-\tfrac{1}{2}\right]\right]\mathrm{Cos}\left[\mathrm{Arg}\left[t\left[1,1,\tfrac{1}{2},-\tfrac{1}{2}\right]\right]-\mathrm{Arg}\left[t\left[2,1,\tfrac{1}{2},-\tfrac{1}{2}\right]\right]\right]\Big)\mathrm{Sin}[\theta]^2\Big)$$

Case 3:$\theta = 0$, Θ, ϕ arbitrary /J =1,2

$4 \times d\sigma/d\Omega =$

$$3\,\mathrm{Abs}[t1[1,0]]^2\,\mathrm{Cos}[\Theta]^2\Big(6\,\mathrm{Abs}\left[t\left[1,0,\tfrac{1}{2},\tfrac{1}{2}\right]\right]^2+25\,\mathrm{Abs}\left[t\left[2,0,\tfrac{1}{2},-\tfrac{1}{2}\right]\right]^2\mathrm{Sin}[\Theta]^2\Big)$$

Observations:
The expression for $d\sigma/d\Omega$ is built up of the same amplitudes and phases as the corresponding expression for $\omega \rightarrow \gamma\pi^0$.Thus,in both cases the measurement of $d\sigma/d\Omega$ yields the same information.

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A Verification of Formula (3)

A mixed elliptical γ state consists of four pure components:
1)Pure state of positive helicity appearing with probability p_+
$|\gamma_1\rangle = |+\rangle$ (see (2) with a=$1/\sqrt{2}$,b=$1/\sqrt{2}$,$\Phi = 0$)
2)Pure state of negative helicity appearing with probability p_-
$|\gamma_2\rangle = |-\rangle$ (see (2) with a= -$1/\sqrt{2}$,b=$1/\sqrt{2}$,$\Phi = 0$)
3)Linearly polarized state with angle Φ to the x-axis appearing with probability $p_{x'}$
$|\gamma_3\rangle = 1/\sqrt{2}(-\exp[-\imath\Phi]|+\rangle + \exp[\imath\Phi]|-\rangle)$ (see (2) with a=1,b=0,$\Phi_{x'} = \Phi$)
4)Linearly polarized state with angle $\Phi_{y'} = \Phi + \pi/2$ to the x-axis appearing with probability $p_{y'}$
$|\gamma_4\rangle = \imath/\sqrt{2}(\exp[-\imath\Phi]|+\rangle + \exp[\imath\Phi]|-\rangle)$ (see (2) with a=1,b=0,$\Phi_{y'} = \Phi + \pi/2$)
with $p_+ + p_- = p_{x'} + p_{y'} = 1$
The corresponding ρ matrices are

$$\rho_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};\ \rho_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\rho_3 = 1/2 \begin{pmatrix} 1 & 0 & -\exp[-2\imath\Phi] \\ 0 & 0 & 0 \\ -\exp[2\imath\Phi] & 0 & 1 \end{pmatrix};\ \rho_4 = 1/2 \begin{pmatrix} 1 & 0 & \exp[-2\imath\Phi] \\ 0 & 0 & 0 \\ \exp[2\imath\Phi] & 0 & 1 \end{pmatrix}$$

With $P_{circ} = p_+ - p_-$ ($p_+ = 1/2(1 + P_{circ}); p_- = 1/2(1 - P_{circ})$) and $P_{lin} = p_{x'} - p_{y'}$.
the ρ matrix of the mixed state is

$$\rho = p_+\rho_1 + p_-\rho_2 + p_{x'}\rho_3 + p_{y'}\rho_4 =$$

$$\rho = 1/2 \begin{pmatrix} 1 + P_{circ} & 0 & -P_{lin}\exp[-2\imath\Phi] \\ 0 & 0 & 0 \\ -P_{lin}\exp[2\imath\Phi] & 0 & 1 - P_{circ} \end{pmatrix}$$